

Supplemental Notes to “Demographic Transition and Industrial Revolution: A Macroeconomic Investigation.”

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1 Derivation of the objective function of the Dynastic Problem (DP)

Proposition 1 *Under the assumption that household utility is $U_t = u(c_t, n_t) + \beta U_{t+1}$, the objective function in (DP) becomes $\sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$. In particular, with $u(c_t, n_t) = \alpha \log c_t + (1 - \alpha) \log n_t$, the objective can be replaced by $\sum_{t=0}^{\infty} \beta^t (\alpha \log C_t + (1 - \alpha - \beta) \log N_{t+1})$.*

Proof. Applying recursive substitution to household utility we obtain

$$\begin{aligned}
U_t &= u(c_t, n_t) + \beta U_{t+1} \\
&= u(c_t, n_t) + \beta (u(c_{t+1}, n_{t+1}) + \beta U_{t+2}) \\
&= u(c_t, n_t) + \beta u(c_{t+1}, n_{t+1}) + \beta^2 (u(c_{t+2}, n_{t+2}) + \beta U_{t+3}) \\
&= u(c_t, n_t) + \beta u(c_{t+1}, n_{t+1}) + \beta^2 u(c_{t+2}, n_{t+2}) + \beta^3 U_{t+3} + \dots \\
&= \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_{\tau}, n_{\tau}) + \lim_{T \rightarrow \infty} \beta^T U_{t+T} = \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_{\tau}, n_{\tau}); \\
U_0 &= \sum_{t=0}^{\infty} \beta^t u(c_t, n_t)
\end{aligned}$$

Substituting for the specific functional form obtains

$$\begin{aligned}
U_0 &= \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) = \sum_{t=0}^{\infty} \beta^t (\alpha \log c_t + (1 - \alpha) \log n_t) = \\
&= \sum_{t=0}^{\infty} \beta^t [\alpha \log C_t + (1 - \alpha) \log N_{t+1} - \log N_t] \\
&= (\alpha \log C_0 + (1 - \alpha) \log N_1 - \log N_0) \\
&\quad + \beta (\alpha \log C_1 + (1 - \alpha) \log N_2 - \log N_1) \\
&\quad + \beta^2 (\alpha \log C_2 + (1 - \alpha) \log N_3 - \log N_2) + \dots \\
&= -\log N_0 + \sum_{t=0}^{\infty} \beta^t \alpha \log C_t + \sum_{t=0}^{\infty} \beta^t (1 - \alpha - \beta) \log N_{t+1} \\
&= -\log N_0 + \sum_{t=0}^{\infty} \beta^t [\alpha \log C_t + (1 - \alpha - \beta) \log N_{t+1}]
\end{aligned}$$

Since N_0 is just a constant, the result holds. ■

When working with this functional form, we assume $1 - \alpha - \beta > 0$ to guarantee the strict concavity of the objective.

2 Characterization of the competitive equilibrium

Next we characterize the Competitive Equilibrium as defined in the main text.

Feasibility and market clearing conditions in the capital, labor, and land markets are given by

$$C_t + K_{t+1} = A_{1t}K_{1t}^\phi L_{1t}^\mu \Lambda_t^{1-\phi-\mu} + A_{2t}K_{2t}^\theta L_{2t}^{1-\theta} + (1-\delta)K_t, \quad (1)$$

$$K_{1t} + K_{2t} = K_t, \quad (2)$$

$$L_{1t} + L_{2t} = (1 - q_t n_t) N_t, \quad (3)$$

$$\Lambda_t = \Lambda. \quad (4)$$

2.1 Dynastic Problem

The dynastic problem (DP) is given by

$$\begin{aligned} & \max_{\{c_t, n_t, \lambda_{t+1}, k_{t+1}\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \\ \text{subject to } c_t + k_{t+1} n_t &= (1 - q_t n_t) w_t + (r_t + 1 - \delta) k_t + \rho_t \lambda_t, \quad \forall t \\ \lambda_{t+1} &= \frac{\lambda_t}{n_t}, \quad c_t, n_t, k_{t+1} \geq 0, \quad k_0, \lambda_0 \text{ given} \end{aligned} \quad (DP)$$

Substituting from the second constraint into the first obtains

$$\begin{aligned} & \max_{\{c_t, n_t, \lambda_{t+1}, k_{t+1}\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \\ \text{subject to } c_t + (k_{t+1} + q_t w_t) n_t &= w_t + (r_t + 1 - \delta) k_t + \rho_t \frac{\lambda_0}{\prod_{\tau=0}^{t-1} n_\tau}, \\ c_t, n_t, k_{t+1} &\geq 0, \quad k_0, \lambda_0 \text{ given.} \end{aligned}$$

Rewriting per household variables in terms of the dynastic aggregates and multiplying the budget constraint by $N_t = N_0 \prod_{\tau=0}^{t-1} n_\tau$, we transform the dynastic problem into (DP') ,

$$\begin{aligned} & \max_{\{C_t, N_{t+1}, K_{t+1}\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t u\left(\frac{C_t}{N_t}, \frac{N_{t+1}}{N_t}\right) \\ \text{subject to } C_t + K_{t+1} &= (N_t - q_t N_{t+1}) w_t + (r_t + 1 - \delta) K_t + \rho_t \Lambda \\ C_t, N_{t+1}, K_{t+1} &\geq 0, \quad K_0, N_0 \text{ given}^1 \end{aligned} \quad (DP') \quad (5)$$

2.1.1 Existence and uniqueness of the solution to (DP)

By construction, (DP) is equivalent to (DP') . The constraint set of (DP') is non-empty and compact and we assume the objective function to be continuous. This guarantees existence of a solution. Since the constraint set is also convex and we assume a strictly concave utility function, the solution is also unique.

2.1.2 Sufficiency of first order and transversality conditions

Next we argue that first order and transversality conditions are sufficient to characterize the unique solution to (DP) .

Let $x_t \equiv (K_t, N_t)$. Then our problem can be written in the form of a sequential problem as in Stokey et al. (1989),

$$\begin{aligned} & \max_{\{x_{t+1}\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}) \\ & \text{s.t.} \\ & x_{t+1} \in \Gamma(x_t), \quad t = 1, 2, \dots, \\ & x_0 \geq 0 \text{ given,} \end{aligned}$$

where

$$\Gamma(K_t, N_t) = \{(K_{t+1}, N_{t+1}) \in \mathcal{R}_+^2 | 0 \leq K_{t+1} \leq (N_t - q_t N_{t+1}) w_t + (r_t + 1 - \delta) K_t + \rho_t \Lambda, \quad 0 \leq N_{t+1} \leq N_t / q_t\}.$$

Under the assumptions of continuity and strict concavity of u , all the conditions of Theorem 4.15 in Stokey, Lucas with Prescott are satisfied and hence the first order and transversality conditions along with the budget constraint can be used to characterize the solution. More precisely for a given x_0 , a sequence $\{x_{t+1}\}_{t=0}^{\infty}$ with $x_{t+1} \in \text{int}\Gamma(x_t)$ for all t solves (DP) if it satisfies first order and transversality conditions. Consumption then is determined from the budget constraint.

We use the first order and transversality conditions for (DP') to characterize the solution to the Dynastic Planning Problem.

2.1.3 Derivation of first order and transversality conditions

Denote the Lagrange multiplier on the time t constraint by φ_t . The first order conditions are given by

$$\begin{aligned} [C_t] &: \beta^t u_1(t) \frac{1}{N_t} = \varphi_t \\ [K_{t+1}] &: \varphi_t = \varphi_{t+1} (r_{t+1} + 1 - \delta), \\ [N_{t+1}] &: \beta^t u_2(t) \frac{1}{N_t} - \beta^{t+1} \left(u_1(t+1) \frac{C_{t+1}}{N_{t+1}^2} + u_2(t+1) \frac{N_{t+2}}{N_{t+1}^2} \right) = \varphi_t q_t w_t - \varphi_{t+1} w_{t+1}. \end{aligned}$$

The first two yield the Euler Equation,

$$\frac{u_1(t)}{u_1(t+1)} = \beta \frac{N_t}{N_{t+1}} (r_{t+1} + 1 - \delta). \quad (6)$$

Dividing the FOC w.r.t. $[N_{t+1}]$ by φ_{t+1} and substituting from the FOC w.r.t. $[C_t]$ gives

$$\begin{aligned} \frac{\beta^t u_2(t) \frac{1}{N_t}}{\beta^{t+1} u_1(t+1) \frac{1}{N_{t+1}}} - \frac{\beta^{t+1} \left(u_1(t+1) \frac{C_{t+1}}{N_{t+1}^2} + u_2(t+1) \frac{N_{t+2}}{N_{t+1}^2} \right)}{\beta^{t+1} u_1(t+1) \frac{1}{N_{t+1}}} &= \frac{\varphi_t}{\varphi_{t+1}} q_t w_t - w_{t+1}, \\ \left(\frac{u_2(t)}{u_1(t)} - q_t w_t \right) (r_{t+1} + 1 - \delta) - \frac{C_{t+1}}{N_{t+1}} &= \frac{u_2(t+1)}{u_1(t+1)} \frac{N_{t+2}}{N_{t+1}} - w_{t+1}. \end{aligned} \quad (7)$$

Hence, the set of conditions describing the solution to the Dynastic Planning problem is given by (6), (7), (DP') and the two transversality conditions

$$\lim_{t \rightarrow \infty} \beta^t u_{K_t} \left(\frac{(N_t - q_t N_{t+1}) w_t + (r_t + 1 - \delta) K_t + \rho_t \Lambda - K_{t+1}}{N_t}, \frac{N_{t+1}}{N_t} \right) K_t = 0 \quad (8)$$

$$\lim_{t \rightarrow \infty} \beta^t u_{N_t} \left(\frac{(N_t - q_t N_{t+1}) w_t + (r_t + 1 - \delta) K_t + \rho_t \Lambda - K_{t+1}}{N_t}, \frac{N_{t+1}}{N_t} \right) N_t = 0 \quad (9)$$

2.2 Firms maximization and optimal resource allocation

2.2.1 The Malthusian technology is always employed

Lemma 2 *The Malthusian technology is always employed in equilibrium.*

Proof. Suppose the Malthusian firm produces $Y_1 = \varepsilon$. The firm's spending on land rent is

$$\rho_t \Lambda = (1 - \phi - \mu) A_1 K_{1t}^\phi L_{1t}^\mu \Lambda^{-\phi-\mu} \cdot \Lambda = (1 - \phi - \mu) \varepsilon.$$

The condition for producing any output at a minimum cost is

$$\frac{\phi A_{1t} K_{1t}^{\phi-1} L_{1t}^\mu \Lambda^{1-\phi-\mu}}{\mu A_{1t} K_{1t}^\phi L_{1t}^{\mu-1} \Lambda^{1-\phi-\mu}} = \frac{r_t}{w_t}, \text{ i.e., } L_{1t} = \left(\frac{r_t \mu}{w_t \phi} \right) K_{1t}.$$

Using this condition and $A_1 K_1^\phi L_1^\mu \Lambda^{1-\phi-\mu} = \varepsilon$ we get the amount of capital employed,

$$A_{1t} K_{1t}^\phi \left(\frac{r_t \mu}{w_t \phi} K_{1t} \right)^\mu \Lambda^{1-\phi-\mu} = \varepsilon,$$

$$K_{1t}^* = \left[\varepsilon A_{1t}^{-1} \left(\frac{r_t \mu}{w_t \phi} \right)^{-\mu} \Lambda^{\phi+\mu-1} \right]^{\frac{1}{\phi+\mu}},$$

which represents the demand for capital conditional on the scale of operation $Y_1 = \varepsilon$. The conditional demand for labor is then given by

$$L_{1t}^* = \left(\frac{r_t \mu}{w_t \phi} \right) \left[\varepsilon A_{1t}^{-1} \left(\frac{r_t \mu}{w_t \phi} \right)^{-\mu} \Lambda^{\phi+\mu-1} \right]^{\frac{1}{\phi+\mu}}.$$

The total cost of producing ε units is therefore

$$r K_{1t}^* + w L_{1t}^* + \rho_t \Lambda = r_t \left[\varepsilon A_{1t}^{-1} \left(\frac{r_t \mu}{w_t \phi} \right)^{-\mu} \Lambda^{\phi+\mu-1} \right]^{\frac{1}{\phi+\mu}} +$$

$$+ w_t \left(\frac{r_t \mu}{w_t \phi} \right) \left[\varepsilon A_{1t}^{-1} \left(\frac{r_t \mu}{w_t \phi} \right)^{-\mu} \Lambda^{\phi+\mu-1} \right]^{\frac{1}{\phi+\mu}} + (1 - \phi - \mu) \varepsilon.$$

We want to show that for ε small enough, the cost of producing ε is lower than ε , i.e.,

$$r_t \left[\varepsilon A_{1t}^{-1} \left(\frac{r_t \mu}{w_t \phi} \right)^{-\mu} \Lambda^{\phi+\mu-1} \right]^{\frac{1}{\phi+\mu}} +$$

$$w_t \left(\frac{r_t \mu}{w_t \phi} \right) \left[\varepsilon A_{1t}^{-1} \left(\frac{r_t \mu}{w_t \phi} \right)^{-\mu} \Lambda^{\phi+\mu-1} \right]^{\frac{1}{\phi+\mu}} + (1 - \phi - \mu) \varepsilon < \varepsilon$$

$$\left(r_t \left[A_{1t}^{-1} \left(\frac{r_t \mu}{w_t \phi} \right)^{-\mu} \Lambda^{\phi+\mu-1} \right]^{\frac{1}{\phi+\mu}} + w_t \left(\frac{r_t \mu}{w_t \phi} \right) \left[A_{1t}^{-1} \left(\frac{r_t \mu}{w_t \phi} \right)^{-\mu} \Lambda^{\phi+\mu-1} \right]^{\frac{1}{\phi+\mu}} \right) \varepsilon^{\frac{1}{\phi+\mu}-1} <$$

$$< (\phi + \mu)$$

For ε small enough, r and w are approximately determined by the Solow sector alone, so they can be treated as given. Since $\phi + \mu < 1$, $\exists \varepsilon > 0$ small enough that the above inequality holds. Hence, the Malthusian technology is always used in equilibrium. ■

Since the Malthusian firm always operates and its profit is maximized, equilibrium factor prices are always given by

$$r_t = \phi A_{1t} K_{1t}^{\phi-1} L_{1t}^\mu \Lambda^{1-\phi-\mu}, \quad (10)$$

$$w_t = \mu A_{1t} K_{1t}^\phi L_{1t}^{\mu-1} \Lambda^{1-\phi-\mu}, \quad (11)$$

$$\rho_t = (1 - \phi - \mu) A_{1t} K_{1t}^\phi L_{1t}^\mu \Lambda^{-\phi-\mu}. \quad (12)$$

2.2.2 Necessary condition for operating the Solow technology

Lemma 3 *Given total resources to be used in production, (K_t, L_t) , the Solow sector operates if*

$$1 \geq \frac{1}{A_{2t}} \left(\frac{\phi A_{1t} K_t^{\phi-1} L_t^\mu \Lambda^{1-\phi-\mu}}{\theta} \right)^\theta \left(\frac{\mu A_{1t} K_t^\phi L_t^{\mu-1} \Lambda^{1-\phi-\mu}}{1-\theta} \right)^{1-\theta}, \quad (13)$$

where the right hand side represents the unit cost of Solow output computed with all resources allocated to the Malthusian sector.

Proof. First we derive the cost of producing ε units of Solow output for some given factor prices? Profit maximization conditions for the Solow firm are $r_t = \frac{\theta A_{2t} K_{2t}^\theta L_{2t}^{1-\theta}}{K_{2t}}$ and $w_t = \frac{(1-\theta) A_{2t} K_{2t}^\theta L_{2t}^{1-\theta}}{L_{2t}}$, and hence factor prices determine the optimal input ratio, $\frac{r_t}{w_t} = \frac{\theta L_{2t}}{(1-\theta) K_{2t}}$. This allows us to determine the optimal employment of each input in production of ε units of output. Solving $A_{2t} K_{2t}^\theta \left(\frac{r_t(1-\theta) K_{2t}}{w_t \theta} \right)^{1-\theta} = \varepsilon$ for K_{2t} gives $K_{2t}^* = \frac{\varepsilon}{A_{2t}} \left(\frac{r_t(1-\theta)}{w_t \theta} \right)^{\theta-1}$ and $L_{2t}^* = \frac{\varepsilon}{A_{2t}} \left(\frac{r_t(1-\theta)}{w_t \theta} \right)^\theta$. Then the cost of producing ε units of output is given by $r_t K_{2t}^* + w_t L_{2t}^* = r_t \frac{\varepsilon}{A_{2t}} \left(\frac{r_t(1-\theta)}{w_t \theta} \right)^{\theta-1} + w_t \frac{\varepsilon}{A_{2t}} \left(\frac{r_t(1-\theta)}{w_t \theta} \right)^\theta = \frac{\varepsilon}{A_{2t}} \left(\frac{r_t}{\theta} \right)^\theta \left(\frac{w_t}{1-\theta} \right)^{1-\theta}$.

Now consider the following equilibrium outcome. For some arbitrary time t , (13) holds, but Solow output is zero. Then all inputs are allocated to the Malthusian sector and profit maximization of the Malthusian sector yields $r_t = \phi A_{1t} K_{1t}^{\phi-1} L_{1t}^\mu \Lambda^{1-\phi-\mu}$ and $w_t = \mu A_{1t} K_{1t}^\phi L_{1t}^{\mu-1} \Lambda^{1-\phi-\mu}$. Producing ε units of Solow output would then cost

$$\frac{\varepsilon}{A_{2t}} \left(\frac{r_t}{\theta} \right)^\theta \left(\frac{w_t}{1-\theta} \right)^{1-\theta} = \frac{\varepsilon}{A_2} \left(\frac{\phi A_{1t} K_t^{\phi-1} L_t^\mu \Lambda^{1-\phi-\mu}}{\theta} \right)^\theta \left(\frac{\mu A_{1t} K_t^\phi L_t^{\mu-1} \Lambda^{1-\phi-\mu}}{1-\theta} \right)^{1-\theta} \leq \varepsilon,$$

where we used (13). It follows that the Solow sector fails to maximize profits, which is a contradiction. ■

When the Solow sector operates, marginal products of capital and labor must equalize across sectors. Hence, given (K_t, L_t) , optimal resource allocation can be summarized as follows

$$\left\{ \begin{array}{l} 0 < K_{1t} < K_t, 0 < L_{1t} < L_t \\ \phi A_{1t} K_{1t}^{\phi-1} L_{1t}^\mu \Lambda^{1-\phi-\mu} = \theta A_{2t} (K_t - K_{1t})^{\theta-1} (L_t - L_{1t})^{1-\theta} \\ \mu A_{1t} K_{1t}^\phi L_{1t}^{\mu-1} \Lambda^{1-\phi-\mu} = (1-\theta) A_{2t} (K_t - K_{1t})^\theta (L_t - L_{1t})^{-\theta} \end{array} \right\} \text{ if (13) holds,} \quad (14)$$

$$K_{1t} = K_t, L_{1t} = L_t \text{ and } K_{2t} = L_{2t} = 0, \quad \text{otherwise.}$$

2.3 Summary of conditions that characterize the competitive equilibrium

Proposition 4 *Conditions (1) – (4), (6) – (12), (14)² characterize (necessary and sufficient conditions for) the solution to the competitive equilibrium.*

Proof. Follows from the definition of the competitive equilibrium and from the above derivations. ■

The following proposition allows us to replace the transversality conditions by a simple check of the limiting behavior of C_t/Y_t and Y_t of the candidate solution.

Proposition 5 *If candidate equilibrium solution sequences of allocations and prices satisfy (1) – (4), (6), (7), (10) – (12), (14), C_t/Y_t is bounded away from 0 and Y_t exhibits growth as $t \rightarrow \infty$, then the transversality conditions (8) and (9) are also satisfied.*

²Note the budget constraint is implied.

Proof. With our choice of the objective function, the transversality conditions for (SP) are given by

$$\begin{aligned}\lim_{t \rightarrow \infty} \beta^t \frac{\alpha(r_t + 1 - \delta)}{(N_t - qN_{t+1})w_t + (r_t + 1 - \delta)K_t + \rho_t \Lambda - K_{t+1}} K_t &= 0 \\ \lim_{t \rightarrow \infty} \beta^t \frac{\alpha w_t}{(N_t - qN_{t+1})w_t + (r_t + 1 - \delta)K_t + \rho_t \Lambda - K_{t+1}} N_t &= 0.\end{aligned}$$

Consider the term inside the first transversality condition

$$\frac{\alpha(r_t + 1 - \delta)}{(N_t - qN_{t+1})w_t + (r_t + 1 - \delta)K_t + \rho_t \Lambda - K_{t+1}} K_t = \frac{\phi Y_{1t} + \theta Y_{2t} + (1 - \delta)K_t}{Y_t + (1 - \delta)K_t - K_{t+1}}$$

Dividing the numerator and the denominator by the level of aggregate output gives

$$\frac{\phi Y_{1t}/Y_t + \theta Y_{2t}/Y_t + (1 - \delta)K_t/Y_t}{1 - (K_{t+1} - (1 - \delta)K_t)/Y_t}$$

The term $K_{t+1} - (1 - \delta)K_t$ represents aggregate investment and $Y_t - (K_{t+1} - (1 - \delta)K_t)$ represents aggregate consumption. Notice that there are two ways to violate this T.V.C. Either the numerator goes to ∞ fast enough or the denominator goes to zero fast enough (or both). Clearly, the numerator cannot go to ∞ since $Y_{1t}/Y_t \in [0, 1]$, $Y_{2t}/Y_t \in [0, 1]$, and $(1 - \delta)K_t/Y_t \rightarrow \infty$ is impossible (Indeed, suppose $(1 - \delta)K_t/Y_t \rightarrow \infty$, then $\delta K_t/Y_t \rightarrow \infty$ and $\exists t^*$ such that $\forall t > t^*$, $K_{t+1} - K_t = X_t - \delta K_t < 0$. This in turn implies that Y_t is shrinking which violates our assumption). The only way the T.V.C. can be violated is when $1 - (K_{t+1} - (1 - \delta)K_t)/Y_t \rightarrow 0$ fast enough, which means that $\frac{C_t}{Y_t}$ goes to zero and again violates our assumption. The argument for the second T.V.C. is similar. ■

Notice that any balanced growth behavior of the equilibrium time paths such that C_t/Y_t is constant guarantees that transversality conditions hold.

3 Sequential Problem (SP) whose solution corresponds to the competitive equilibrium allocation

3.1 Definition

We next define a sequential problem whose solution is the competitive equilibrium allocation. This sequential problem (SP) compactly states the optimization problem at hand and illustrates the sense in which the competitive equilibrium allocation is efficient. There are difficulties associated with defining efficiency in models with endogenous fertility. The (SP) defined here corresponds to the \mathcal{A} -efficiency concept as defined by Golosov, Jones, Tertilt (2006). According to this concept, when comparisons are made across allocations, the positive weight is put only on those households that are alive in all possible allocations. Analyzing concepts of efficiency in models of endogenous fertility, however, is beyond the scope of this paper.

We emphasize that proving equivalence of the competitive equilibrium allocation to the solution of this problem is not necessary for any of the results in the main text. To solve the model, it is enough to use sufficient conditions derived in the previous section. What we do in this section is driven strictly by intellectual curiosity.

$$\begin{aligned}\max_{\{C_t, N_{t+1}, K_{t+1}\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t u\left(\frac{C_t}{N_t}, \frac{N_{t+1}}{N_t}\right) \\ \text{subject to } C_t + K_{t+1} &= F(K_t, L_t; t) + (1 - \delta)K_t, \\ L_t &= N_t - q_t N_{t+1}, \\ C_t, K_{t+1}, N_{t+1} &> 0, \quad K_0, N_0 \text{ are given and}\end{aligned} \tag{15}$$

$$\begin{aligned}F(K_t, L_t; t) &= \max_{K_{1t}, L_{1t}} \left[A_{1t} K_{1t}^\phi L_{1t}^\mu \Lambda^{1-\phi-\mu} + A_{2t} (K_t - K_{1t})^\theta (L_t - L_{1t})^{1-\theta} \right] \\ \text{subject to } 0 &\leq K_{1t} \leq K_t, \quad 0 \leq L_{1t} \leq L_t.\end{aligned} \tag{16}$$

3.1.1 Existence and uniqueness of solution to (SP)

Continuity of the objective function in (SP), together with the compactness and non-emptiness of the constraint set, guarantees existence of a solution. We assume strict concavity of u which guarantees that the objective function is strictly concave. Since the constraint set is convex, the solution is unique.

3.1.2 Sufficiency of first order and transversality conditions

Next we argue that first order and transversality conditions are sufficient to characterize the unique solution to (SP). We follow the lines of the argument put forth for the dynastic problem, except that now

$$\Gamma(K_t, N_t) = \{(K_{t+1}, N_{t+1}) \in \mathcal{R}_+^2 | 0 \leq K_{t+1} \leq F(K_t, N_t - q_t N_{t+1}) + (1 - \delta)K_t, 0 \leq N_{t+1} \leq N_t/q_t\}$$

The set $\Gamma(K_t, N_t)$ is illustrated graphically in Figure 1. Notice that the frontier of the set is given by

$$K_{t+1} = F(K_t, N_t - q_t N_{t+1}) + (1 - \delta)K_t$$

and we have

$$\begin{aligned} \frac{dK_{t+1}}{dN_{t+1}} &= -q_t F_2(K_t, N_t - q_t N_{t+1}) < 0 \\ \frac{d^2 K_{t+1}}{dN_{t+1}^2} &= q_t^2 F_{22}(K_t, N_t - q_t N_{t+1}) < 0 \end{aligned}$$

Thus, the frontier of this set is strictly decreasing and a strictly concave function, so the set $\Gamma(K_t, N_t)$ is convex.

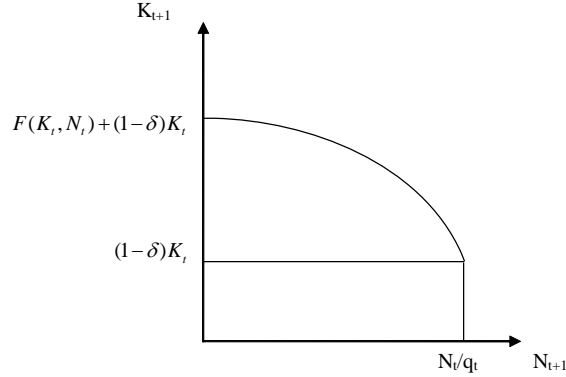


Figure 1. Constraint Set at time t

Under the assumptions of continuity and strict concavity of u , all the conditions of Theorem 4.15 in Stokey, Lucas with Prescott are satisfied. Given x_0 , a sequence $\{x_{t+1}\}_{t=0}^\infty$ with $x_{t+1} \in \text{int}\Gamma(x_t)$ for all t solves (DP) if it satisfies first order and transversality conditions. Consumption then is determined from (15) while Kuhn-Tucker conditions determine optimal resource allocation.

3.1.3 Derivation of first order and transversality conditions

Denoting the multiplier on (15) by μ_t , we obtain the following first order conditions:

$$\begin{aligned} [C_t] &: \beta^t u_1(t) \frac{1}{N_t} = \mu_t, \\ [K_{t+1}] &: \mu_t = \mu_{t+1} [F_1(K_{t+1}, N_{t+1} - q_{t+1} N_{t+2}; t+1) + 1 - \delta], \\ [N_{t+1}] &: \beta^t u_2(t) \frac{1}{N_t} - \beta^{t+1} \left(u_1(t+1) \frac{C_{t+1}}{N_{t+1}^2} + u_2(t+1) \frac{N_{t+2}}{N_{t+1}^2} \right) = \\ &= \mu_t q_t F_2(K_t, N_t - q_t N_{t+1}; t) - \mu_{t+1} F_2(K_{t+1}, N_{t+1} - q_{t+1} N_{t+2}; t+1) \end{aligned}$$

The first two conditions yield

$$\frac{u_1(t)}{u_1(t+1)} = \beta \frac{N_t}{N_{t+1}} (F_1(K_{t+1}, N_{t+1} - q_{t+1}N_{t+2}; t+1) + 1 - \delta). \quad (17)$$

Dividing the FOC w.r.t. N_{t+1} by μ_{t+1} and substituting from the FOC w.r.t. C_t for μ_t and μ_{t+1} gives

$$\begin{aligned} & \frac{\beta^t u_2(t) \frac{1}{N_t}}{\beta^{t+1} u_1(t+1) \frac{1}{N_{t+1}}} - \frac{\beta^{t+1} \left(u_1(t+1) \frac{C_{t+1}}{N_{t+1}^2} + u_2(t+1) \frac{N_{t+2}}{N_{t+1}^2} \right)}{\beta^{t+1} u_1(t+1) \frac{1}{N_{t+1}}} = \\ & = \frac{\mu_t}{\mu_{t+1}} q_t F_2(K_t, N_t - q_t N_{t+1}; t) - F_2(K_{t+1}, N_{t+1} - q_{t+1} N_{t+2}; t+1), \\ \text{i.e., } & \left(\frac{u_2(t)}{u_1(t)} - q_t w_t \right) (F_1(K_{t+1}, N_{t+1} - q_{t+1} N_{t+2}; t+1) + 1 - \delta) - \frac{C_{t+1}}{N_{t+1}} = \\ & = \frac{u_2(t+1)}{u_1(t+1)} \frac{N_{t+2}}{N_{t+1}} - F_2(K_{t+1}, N_{t+1} - q_{t+1} N_{t+2}; t+1). \end{aligned} \quad (18)$$

The transversality conditions for (SP) are

$$\begin{aligned} \lim_{t \rightarrow \infty} \beta^t u_{K_t} \left(\frac{F(K_t, N_t - q_t N_{t+1}; t) + (1 - \delta)K_t - K_{t+1}}{N_t}, \frac{N_{t+1}}{N_t} \right) K_t &= 0, \\ \lim_{t \rightarrow \infty} \beta^t u_{N_t} \left(\frac{F(K_t, N_t - q_t N_{t+1}; t) + (1 - \delta)K_t - K_{t+1}}{N_t}, \frac{N_{t+1}}{N_t} \right) N_t &= 0. \end{aligned}$$

The static maximization problem (16) determines the optimal resource allocation between the two technologies. In the following Lemma we show that the Malthusian output is always strictly positive ($K_{1t}, L_{1t} > 0 \forall t$).

3.1.4 Optimal resource allocation

Lemma 6 *In any solution to the Sequential Problem, the Malthusian technology operates for all t as long as $K_t, L_t, A_{1t} > 0$.*

Proof. Suppose on the contrary that there is time t such that $Y_{1t} = 0$. Since resources are allocated efficiently, this means that $K_{1t} = L_{1t} = 0$ and

$$\max_{K_t, L_{1t}} \left[A_{1t} K_{1t}^\phi L_{1t}^\mu \Lambda^{1-\phi-\mu} + A_{2t} (K_t - K_{1t})^\theta (L_t - L_{1t})^{1-\theta} \right] = A_{2t} K_t^\theta L_t^{1-\theta}. \quad (19)$$

Consider reallocating $(\varepsilon K_t, \varepsilon L_t)$ to the Malthusian technology, where $\varepsilon \in (0, 1)$. Next we show that for ε small enough,

$$A_{1t} (\varepsilon K_t)^\phi (\varepsilon L_t)^\mu \Lambda^{1-\phi-\mu} + A_{2t} ((1 - \varepsilon) K_t)^\theta ((1 - \varepsilon) L_t)^{1-\theta} > A_{2t} K_t^\theta L_t^{1-\theta}. \quad (20)$$

Simplifying this inequality gives $\frac{1}{\varepsilon^{1-\phi-\mu}} > \frac{A_{2t} K_t^\theta L_t^{1-\theta}}{A_{1t} K_t^\phi L_t^\mu \Lambda^{1-\phi-\mu}}$. Since $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{1-\phi-\mu}} = \infty$ and the right hand side is a finite number, $\exists \varepsilon > 0$ that ensures (20) is satisfied. Hence, we arrive at contradiction with (19). ■

Next we derive conditions that characterize the solution to (16). To ease notation, denote $N_t - q_t N_{t+1}$ by L_t and ignore the time subscript since this maximization problem is static. The only possible corner solution here is the Solow technology not being used.

The Kuhn-Tucker conditions for (16) are given by

$$\begin{aligned} \phi A_1 K_1^{\phi-1} L_1^\mu \Lambda^{1-\phi-\mu} - \theta A_2 (K - K_1)^{\theta-1} (L - L_1)^{1-\theta} &\geq 0, \\ &= \text{if } K_1 < K; \\ \mu A_1 K_1^\phi L_1^{\mu-1} \Lambda^{1-\phi-\mu} - (1 - \theta) A_2 (K - K_1)^\theta (L - L_1)^{-\theta} &\geq 0, \\ &= \text{if } L_1 < L. \end{aligned}$$

When the Solow technology operates,

$$\begin{aligned} \frac{\theta A_2 (K - K_1)^{\theta-1} (L - L_1)^{1-\theta}}{(1-\theta) A_2 (K - K_1)^\theta (L - L_1)^{-\theta}} &= \frac{\phi A_1 K_1^{\phi-1} L_1^\mu \Lambda^{1-\phi-\mu}}{\mu A_1 K_1^\phi L_1^{\mu-1} \Lambda^{1-\phi-\mu}}, \\ \text{i.e., } \frac{\theta}{1-\theta} \frac{L_2}{K_2} &= \frac{\phi A_1 K_1^{\phi-1} L_1^\mu \Lambda^{1-\phi-\mu}}{\mu A_1 K_1^\phi L_1^{\mu-1} \Lambda^{1-\phi-\mu}}. \end{aligned}$$

Hence,

$$\lim_{K_1 \rightarrow K, L_1 \rightarrow L} \frac{\theta}{1-\theta} \frac{L_2}{K_2} = \frac{\phi A_1 K^{\phi-1} L^\mu \Lambda^{1-\phi-\mu}}{\mu A_1 K^\phi L^{\mu-1} \Lambda^{1-\phi-\mu}}.$$

Thus the Solow technology does not operate whenever

$$\begin{aligned} \phi A_1 K_1^{\phi-1} L_1^\mu \Lambda^{1-\phi-\mu} - \theta A_2 (K - K_1)^{\theta-1} (L - L_1)^{1-\theta} &> 0 \\ \phi A_1 K^{\phi-1} L^\mu \Lambda^{1-\phi-\mu} - \lim_{K_1 \rightarrow K, L_1 \rightarrow L} \theta A_2 \left(\frac{L_2}{K_2} \right)^{1-\theta} &> 0 \\ \phi A_1 K^{\phi-1} L^\mu \Lambda^{1-\phi-\mu} - \theta A_2 \left(\frac{1-\theta}{\theta} \frac{\phi A_1 K^{\phi-1} L^\mu \Lambda^{1-\phi-\mu}}{\mu A_1 K^\phi L^{\mu-1} \Lambda^{1-\phi-\mu}} \right)^{1-\theta} &> 0 \\ \left(\frac{\phi A_1 K^{\phi-1} L^\mu \Lambda^{1-\phi-\mu}}{\theta} \right)^\theta \left(\frac{\mu A_1 K^\phi L^{\mu-1} \Lambda^{1-\phi-\mu}}{1-\theta} \right)^{1-\theta} \frac{1}{A_2} &> 1 \end{aligned} \quad (21)$$

Given K and L , the optimal resource allocation can be summarized as follows

$$\begin{aligned} K_1 = K, L_1 = L, K_2 = L_2 = 0, & \quad \text{if (21) holds} \\ \left\{ \begin{array}{l} 0 < K_1 < K, 0 < L_1 < L, \text{ and} \\ \phi A_1 K_1^{\phi-1} L_1^\mu \Lambda^{1-\phi-\mu} = \theta A_2 (K - K_1)^{\theta-1} (L - L_1)^{1-\theta}, \\ \mu A_1 K_1^\phi L_1^{\mu-1} \Lambda^{1-\phi-\mu} = (1-\theta) A_2 (K - K_1)^\theta (L - L_1)^{-\theta} \end{array} \right. & \quad \text{otherwise.} \end{aligned} \quad (22)$$

Since some resources are always allocated to the Malthusian technology, we use

$$\begin{aligned} F_1(K_t, N_t - q_t N_{t+1}; t) &= \phi A_{1t} K_{1t}^{\phi-1} L_{1t}^\mu \Lambda^{1-\phi-\mu} \text{ and} \\ F_2(K_t, N_t - q_t N_{t+1}; t) &= \mu A_{1t} K_{1t}^\phi L_{1t}^{\mu-1} \Lambda^{1-\phi-\mu} \end{aligned}$$

in (17) and (18).

3.1.5 Equivalence

Proposition 7 *The competitive equilibrium in the decentralized economy corresponds to the solution of (SP).*

Proof. Conditions sufficient to determine the unique competitive equilibrium allocation coincide with sufficient conditions for the unique solution to (SP). Hence, the result follows. ■

4 Limiting behavior of the equilibrium time paths

Our goal here is to understand how the behavior of the solution to the model depends on the choice of the parameters and the initial conditions. Denote the choice of the parameters $(\Lambda, A_{10}, A_{20}, \pi, \delta, \gamma_1, \gamma_2, \mu, \phi, \theta, \beta, a, b, \alpha)$ by $\hat{\theta} \in \Theta$, where Θ denotes the set of all admissible parameter choices.

We can identify three possible types of limiting behavior of the equilibrium time paths: (1) The solution exhibits the property that the level of output in both sectors converges to some constant positive fraction of total output, (2) The solution exhibits the property that the level of output in the Solow sector converges to 0, (3) The solution exhibits the property that the level of output in the Malthusian sector relative to

the total output converges to 0. We refer to these types of limiting behavior of equilibrium time paths as convergence to Malthus-Solow Balanced Growth Path (BGP), Malthus BGP and Solow BGP respectively. There are possibly other types of behavior of equilibrium time paths but we do not attempt to describe those here.

Define the following sets of parameter and initial condition values that generate one of the three types of limiting behavior of the equilibrium paths described above:

$$\begin{aligned} S_1 &= \left\{ \hat{\theta} \in \Theta, (k_0, N_0) \in \mathcal{R}_+^2 \mid \lim_{t \rightarrow \infty} \frac{y_{1t}(\hat{\theta}, k_0, N_0)}{y_t(\hat{\theta}, k_0, N_0)} = \rho_y \in (0, 1) \right\}, \\ S_2 &= \left\{ \hat{\theta} \in \Theta, (k_0, N_0) \in \mathcal{R}_+^2 \mid \lim_{t \rightarrow \infty} y_{2t}(\hat{\theta}, k_0, N_0) = 0 \right\}, \\ S_3 &= \left\{ \hat{\theta} \in \Theta, (k_0, N_0) \in \mathcal{R}_+^2 \mid \lim_{t \rightarrow \infty} \frac{y_{1t}(\hat{\theta}, k_0, N_0)}{y_t(\hat{\theta}, k_0, N_0)} = 0 \right\}. \end{aligned}$$

The objective is to describe S_1, S_2 and S_3 as best we can. We can also identify two subsets of S_1 and S_2 that we index by “*”:

$$\begin{aligned} S_1^* &= \left\{ \hat{\theta} \in \Theta, (k_0, N_0) \in \mathcal{R}_+^2 \mid \forall t, \frac{y_{1t}(\hat{\theta}, k_0, N_0)}{y_t(\hat{\theta}, k_0, N_0)} = \rho_y \in (0, 1) \right\}, \\ S_2^* &= \left\{ \hat{\theta} \in \Theta, (k_0, N_0) \in \mathcal{R}_+^2 \mid \forall t, y_{2t}(\hat{\theta}, k_0, N_0) = 0 \right\}. \end{aligned}$$

If the parameter values and initial conditions lie in S_1^* , then the equilibrium time paths are on a Malthus-Solow BGP starting in period 0. Similarly, if the parameter values and initial conditions lie in S_2^* , then the equilibrium time paths are on a Malthus BGP starting in period 0. Note that a similar subset does not exist in S_3 because the Malthusian technology always operates as we proved in Lemmas 2 and 5.

4.1 Propositions summarizing how the behavior of equilibrium time paths depends on the choice of parameters and initial conditions

We show in detail derivations upon which the results that we present here rest and summarize these as lemmas and propositions. We summarize our findings regarding the dependence of the solution behavior on the choice of parameters and conditions. We discuss our results and illustrate a stylized segmentation of the parameter and initial condition space into different types of limiting equilibrium behavior. Our discussion will contrast the result obtained by Hansen and Prescott (2003). In that work, as long as the growth rate of the Solow TFP is positive, all equilibria exhibit convergence to a Solow BGP. In our model, however, the limiting behavior of equilibrium time paths is determined by the particular parameterization and initial conditions.

4.1.1 Derivation and discussion of Malthus-Solow balanced growth properties

Next we derive the properties of the limiting behavior of equilibrium time paths in S_1 . We consider equilibrium paths along which all variables grow at constant rates, although not necessarily the same, and the relative output of the two sectors remains constant. The constancy of the growth rates allows us to simplify the relevant equilibrium conditions and arrive at equilibrium growth rates of such a solution as functions of parameters.

Proposition 8 *If there is a solution $\{c_t, N_t, k_t, k_{1t}, y_t, y_{1t}\}_{t=0}^\infty$ such that all variables grow at constant rates $\forall t$, say $\gamma_c, n, \gamma_k, \gamma_{k1}, \gamma_y, \gamma_{y1} > 0$ and $y_{1t}/y_t \equiv \rho_y \in (0, 1)$, then the following is true.*

(1) $\gamma_c = \gamma_k = \gamma_{k1} = \gamma_y = \gamma_{y1} \equiv \gamma$.

(2) The unknowns γ , n , r , l_1 , ρ , ρ_k , ρ_y (where $\rho = \frac{c}{k}$, $\rho_k = \frac{k_1}{k}$, $\rho_y = \frac{y_1}{y}$) satisfy the following equations,³

$$\gamma = \gamma_2^{\frac{1}{1-\theta}}, \quad (23)$$

$$n = \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}} \right)^{\frac{1}{1-\phi-\mu}}, \quad (24)$$

$$\gamma = \frac{\beta}{n} [r + 1 - \delta], \quad (25)$$

$$\frac{(1-\alpha-\beta)}{\alpha n} \frac{\rho \phi l_1}{\mu r \rho_k} = q - \frac{\gamma}{(r+1-\delta)}, \quad (26)$$

$$\frac{\theta \rho_k}{(1-\rho_k)} = \frac{\phi \rho_y}{(1-\rho_y)}, \quad (27)$$

$$\frac{\mu \rho_y}{(1-\rho_y)} = \frac{(1-\theta) l_1}{(1-l_1-qn)}, \quad (28)$$

$$\rho + \gamma n = \frac{r \rho_k}{\rho_y \phi} + (1-\delta). \quad (29)$$

(3) Corresponding efficiency variables, defined as follows,

$$c_t^* = \frac{c_t}{\gamma^t}, \quad k_t^* = \frac{k_t}{\gamma^t}, \quad k_{1t}^* = \frac{k_{1t}}{\gamma^t}, \quad l_{1t}^* = l_{1t}, \quad N_t^* = \frac{N_t}{n^t}, \quad w_{1t}^* = \frac{w_{1t}}{\gamma^t}, \quad r_t^* = r_t,$$

with γ and n given by (23) and (24), are in steady state (which we denote by a bar) and satisfy (30) – (35), given below,

$$\bar{r} \equiv \frac{\gamma n}{\beta} - (1-\delta) > 0, \quad (30)$$

$$\bar{k} \equiv \frac{\left(\frac{\alpha(qn-\beta)}{(1-\alpha-\beta)} \frac{(1-\theta)\bar{r}}{\theta\chi} + \frac{(1-qn)\zeta\bar{r}}{\chi} \left(\frac{1}{\theta} - \frac{1}{\phi} \right) \right)}{\left(\frac{(1+\zeta)\bar{r}}{\theta} - \frac{\bar{r}}{\phi}\zeta + 1 - \delta - \gamma n \right)} > 0 \quad (31)$$

$$0 < \bar{k}_1 \equiv \zeta \left(\frac{1-qn}{\chi} - \bar{k} \right) < \bar{k}, \quad (32)$$

$$0 < \bar{l}_1 \equiv \frac{\mu\theta}{\phi(1-\theta)} \chi \bar{k}_1 < 1-qn, \quad (33)$$

$$\bar{c} \equiv \frac{\alpha}{(1-\alpha-\beta)} (qn-\beta) \frac{\bar{r}\mu\bar{k}_1}{\phi\bar{l}_1} > 0, \quad (34)$$

$$\bar{N} \equiv \left(\frac{\phi A_{10} \bar{k}_1^{\phi-1} \bar{l}_1^\mu}{\bar{r}} \right)^{\frac{1}{1-\phi-\mu}} \Lambda > 0, \quad (35)$$

$$\text{where } \chi \equiv \left(\frac{\bar{r}}{\theta A_{20}} \right)^{\frac{1}{1-\theta}} \text{ and } \zeta \equiv \frac{\phi(1-\theta)}{\mu\theta - \phi(1-\theta)}.$$

(4) Initial conditions (k_0, N_0) generating such a solution correspond to (\bar{k}, \bar{N}) .

³There is a unique analytical solution to this system of equations, which is derived in the proof.

Proof. Since both sectors always operate in the proposed solution, it must satisfy the following necessary equilibrium conditions,

$$\frac{c_{t+1}}{c_t} = \frac{\beta}{n_t} (r_{t+1} + 1 - \delta), \quad (36)$$

$$\frac{(1 - \alpha - \beta) c_t}{\alpha n_t} = q w_t - \frac{w_{t+1}}{(r_{t+1} + 1 - \delta)}, \quad (37)$$

$$\phi A_{10} \gamma_1^t k_{1t}^{\phi-1} l_{1t}^\mu \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu} = \theta A_{20} \gamma_2^t (k_t - k_{1t})^{\theta-1} (1 - l_{1t} - q n_t)^{1-\theta} = r_t, \quad (38)$$

$$\mu A_{10} \gamma_1^t k_{1t}^\phi l_{1t}^{\mu-1} \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu} = (1 - \theta) A_{20} \gamma_2^t (k_t - k_{1t})^\theta (1 - l_{1t} - q n_t)^{-\theta} = w_t, \quad (39)$$

$$c_t + k_{t+1} n_t = A_{10} \gamma_1^t k_{1t}^\phi l_{1t}^\mu \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu} + A_{20} \gamma_2^t (k_t - k_{1t})^\theta (1 - l_{1t} - q n_t)^{1-\theta} + (1 - \delta) k_t, \quad (40)$$

where $n_t = \frac{N_{t+1}}{N_t}$ and (38) and (39) represent equality of factor marginal products across the two sectors.

Denote the constant rate of growth of per capita consumption by γ and the constant growth rate of population by n . From the first equation, which becomes

$$\gamma = \frac{\beta}{n} [r_{t+1} + 1 - \delta]$$

on a BGP, we see that r_t must remain constant, so we replace it by r . Then constancy of r together with its definition in (38) imply that on such a BGP, $y_{1t} = A_{10} \gamma_1^t k_{1t}^\phi l_{1t}^\mu \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu}$ grows at the same rate as k_{1t} while $y_{2t} = A_{20} \gamma_2^t (k_t - k_{1t})^\theta (1 - l_{1t} - q n)^{1-\theta}$ grows at the same rate as $(k_t - k_{1t})$. Since y_{2t} grows at a constant rate by assumption, it means that k_t and k_{1t} grow at the same rate, hence $\rho_k = k_{1t}/k_t$ must stay constant. Once again considering the definition of r in (38), $\theta A_{20} \gamma_2^t (k_t - k_{1t})^{\theta-1} (1 - l_{1t} - q n)^{1-\theta} = r$, we observe that since $k_t - k_{1t}$ grows at a constant rate, $1 - l_{1t} - q n$ must also grow at a constant rate, and since $q n$ and 1 are both constants, it implies that l_{1t} must also stay constant, denote it by l_1 .

Next we show that c_t and k_t must grow at the same constant rate. Consider (40) rewritten using the definition of r from (38)

$$\begin{aligned} c_t + k_{t+1} n_t &= \frac{r k_{1t}}{\phi} + \frac{r (k_t - k_{1t})}{\theta} + (1 - \delta) k_t, \\ \frac{c_t}{k_t} + \frac{k_{t+1}}{k_t} n &= \frac{r k_{1t}}{\phi k_t} + \frac{r (k_t - k_{1t})}{\theta k_t} + (1 - \delta), \\ \frac{c_t}{k_t} &= \frac{r}{\phi} \rho_k + \frac{r}{\theta} (1 - \rho_k) + (1 - \delta) - \frac{k_{t+1}}{k_t} n. \end{aligned}$$

The right hand side is a constant, hence, the left hand side must also remain constant, denote it by $\rho = \frac{c_t}{k_t}$. This means that indeed c_t and k_t grow at the same rate γ . Considering our previous results, this means that y_{1t} and y_{2t} also grow at the rate of γ . Define the fraction $\rho_y = \frac{y_{1t}}{y_t}$.

We can find γ by once again using the definition of r in (38), and its constancy,

$$\theta A_{20} \gamma_2^t (k_t - \rho_k k_t)^{\theta-1} (1 - l_1 - q n)^{1-\theta} = r, \quad (41)$$

$$\gamma = \gamma_2^{\frac{1}{1-\theta}}.$$

Using the definition of r in (38) again but this time as a marginal product in the Malthusian sector, $y_{1t} = A_{10} \gamma_1^t k_{1t}^\phi l_{1t}^\mu \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu}$, we find the relationship between γ and n ,

$$\gamma_1 = n^{1-\phi-\mu} \gamma^{1-\phi}. \quad (42)$$

These last two equations, (41) and (42), pin down the growth rate of per capita variables γ and the population growth rate n precisely

$$\gamma = \gamma_2^{\frac{1}{1-\theta}}, \quad n = \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}} \right)^{\frac{1}{1-\phi-\mu}}.$$

The Malthusian output can be rewritten as $y_{1t} = A_{10} \gamma_1^t k_{1t}^\phi l_1^\mu \lambda_t^{1-\phi-\mu} = \frac{r \rho_k k_t}{\phi}$ or as $y_{1t} = A_{10} \gamma_1^t k_{1t}^\phi l_1^\mu \lambda_t^{1-\phi-\mu} = \frac{w_t l_1}{\mu}$. Hence, we can solve for w_t in terms of k_t :

$$\begin{aligned} \frac{r \rho_k k_t}{\phi} &= \frac{w_t l_1}{\mu}, \\ w_t &= \frac{\mu r \rho_k k_t}{\phi l_1}. \end{aligned}$$

Notice that this implies that w_t also grows at the rate of γ and

$$\frac{c_t}{w_t} = \frac{c_t \phi l_1}{\mu r \rho_k k_t} = \frac{\rho \phi l_1}{\mu r \rho_k}.$$

This allows us to rewrite (37) as follows,

$$\begin{aligned} \frac{(1-\alpha-\beta) c_t}{\alpha n w_t} &= q - \frac{\gamma}{(r+1-\delta)} \\ \frac{(1-\alpha-\beta) \rho \phi l_1}{\alpha n \mu r \rho_k} &= q - \frac{\gamma}{(r+1-\delta)} \end{aligned}$$

Equation (38) can be written as

$$\begin{aligned} \frac{\phi y_{1t}}{\rho_k k_t} &= \frac{\theta y_{2t}}{(1-\rho_k) k_t}, \\ \frac{\phi \rho_y y_t}{\rho_k} &= \frac{\theta (1-\rho_y) y_t}{(1-\rho_k)}, \\ \frac{\phi \rho_y}{(1-\rho_y)} &= \frac{\theta \rho_k}{(1-\rho_k)}. \end{aligned}$$

and equation (39) as

$$\begin{aligned} \frac{\mu y_{1t}}{l_1} &= \frac{(1-\theta) y_{2t}}{(1-l_1-qn)}, \\ \frac{\mu \rho_y}{(1-\rho_y)} &= \frac{(1-\theta) l_1}{(1-l_1-qn)}. \end{aligned}$$

Finally, we rewrite the feasibility condition (40) as

$$\begin{aligned} c_t + k_{t+1} n &= y_t + (1-\delta) k_t, \\ c_t + k_{t+1} n &= \frac{y_{1t}}{\rho_y} + (1-\delta) k_t, \\ c_t + k_{t+1} n &= \frac{r \rho_k k_t}{\rho_y \phi} + (1-\delta) k_t, \\ \rho + \gamma n &= \frac{r \rho_k}{\rho_y \phi} + (1-\delta). \end{aligned}$$

Hence, we proved results (1) and (2) of the lemma.

In fact, we can find the solution to (23) – (29) analytically. The first two equations give γ and n in terms of parameter values. Equation (25) gives

$$r = \frac{\gamma n}{\beta} - (1-\delta).$$

We use (27) to solve for ρ_y in terms of ρ_k

$$\rho_y = \frac{\theta \rho_k}{\phi(1 - \rho_k) + \theta \rho_k}. \quad (43)$$

Substituting the above expression (43) into (28) and then solving for l_1 in terms of ρ_k gives

$$\begin{aligned} \frac{\mu \frac{\theta \rho_k}{\phi(1 - \rho_k) + \theta \rho_k}}{\left(1 - \frac{\theta \rho_k}{\phi(1 - \rho_k) + \theta \rho_k}\right)} &= \frac{(1 - \theta) l_1}{(1 - l_1 - qn)}, \\ \frac{\mu \theta \rho_k}{\phi(1 - \rho_k) + \theta \rho_k - \theta \rho_k} &= \frac{(1 - \theta) l_1}{(1 - l_1 - qn)}, \\ l_1 &= \frac{\mu \theta \rho_k (1 - qn)}{(1 - \theta) \phi(1 - \rho_k) + \mu \theta \rho_k}. \end{aligned} \quad (44)$$

Substituting from (43) into (29) gives ρ in terms of ρ_k ,

$$\rho = \frac{r(\phi(1 - \rho_k) + \theta \rho_k)}{\phi \theta} - \gamma n + (1 - \delta). \quad (45)$$

Finally, substituting from (45) and (44) into (26) gives

$$\begin{aligned} \frac{(1 - \alpha - \beta)}{\alpha n} \phi \left(\frac{r(\phi(1 - \rho_k) + \theta \rho_k)}{\phi \theta} - \gamma n + (1 - \delta) \right) \frac{\mu \theta \rho_k (1 - qn)}{\mu r \rho_k (1 - \theta) \phi(1 - \rho_k) + \mu \theta \rho_k} &= q - \frac{\gamma}{(r + 1 - \delta)}, \\ \frac{(1 - \alpha - \beta)}{\alpha n} (r(\phi(1 - \rho_k) + \theta \rho_k) + \phi \theta (1 - \delta - \gamma n)) (1 - qn) &= r \left(q - \frac{\gamma}{r + 1 - \delta} \right) ((1 - \theta) \phi(1 - \rho_k) + \mu \theta \rho_k), \\ \rho_k &= \phi \frac{r \left(q - \frac{\gamma}{r + 1 - \delta} \right) (1 - \theta) + \frac{(1 - \alpha - \beta)}{\alpha n} (1 - qn) (\theta(\gamma n - (1 - \delta)) - r)}{r \left(q - \frac{\gamma}{r + 1 - \delta} \right) ((1 - \theta) \phi - \mu \theta) + \frac{(1 - \alpha - \beta)}{\alpha n} (1 - qn) r (\theta - \phi)}. \end{aligned}$$

We already derived r, γ, n in terms of parameter values, hence, the above gives the analytical solution for ρ_k along a MS BGP. We can then back out the rest of the variables using (43), (44) and (45).

Recall that equations (36) – (40) characterize the equilibrium with both technologies operating. Rewriting these in terms of efficiency variables defined in the statement of the proposition obtains

$$\begin{aligned} \frac{c_{t+1}^* \gamma}{c_t^*} &= \beta \frac{N_t^*}{N_{t+1}^* n} \left(\phi A_{10} k_{1t+1}^{*\phi-1} l_{1t+1}^{*\mu} \left(\frac{\Lambda}{N_{t+1}^*} \right)^{1-\phi-\mu} + 1 - \delta \right), \\ \frac{(1 - \alpha - \beta) c_t^* N_t^*}{\alpha N_{t+1}^* n} &= q_t w_t^* - \frac{w_{t+1}^* \gamma}{(r_{t+1}^* + 1 - \delta)}, \\ \phi A_{10} k_{1t}^{*\phi-1} l_{1t}^{*\mu} \left(\frac{\Lambda}{N_t^*} \right)^{1-\phi-\mu} &= \theta A_{20} (k_t^* - k_{1t}^*)^{\theta-1} \left(1 - l_{1t}^* - q \frac{N_{t+1}^* n}{N_t^*} \right)^{1-\theta} = r_t^*, \\ \mu A_{10} k_{1t}^{*\phi} l_{1t}^{*\mu-1} \left(\frac{\Lambda}{N_t^*} \right)^{1-\phi-\mu} &= (1 - \theta) A_{20} (k_t^* - k_{1t}^*)^\theta \left(1 - l_{1t}^* - q \frac{N_{t+1}^* n}{N_t^*} \right)^{-\theta} = w_t^*, \\ c_t^* + k_{t+1}^* \gamma \frac{N_{t+1}^* n}{N_t^*} &= \\ &= A_{10} k_{1t}^{*\phi} l_{1t}^{*\mu} \left(\frac{\Lambda}{N_t^*} \right)^{1-\phi-\mu} + A_{20} (k_t^* - k_{1t}^*)^\theta \left(1 - l_{1t}^* - q \frac{N_{t+1}^* n}{N_t^*} \right)^{1-\theta} + (1 - \delta) k_t^*. \end{aligned}$$

Whenever the original variables exhibit Mathus-Solow balanced growth, the efficiency variables are in steady state. This is true by construction of efficiency variables (that utilized information on γ and n along a

Malthus-Solow BGP). It is possible to solve for the steady state values of efficiency variables, which we denote by a bar, analytically. The solution is obtained by solving

$$\gamma n = \beta [\bar{r} + 1 - \delta] \quad (46)$$

$$\frac{(1 - \alpha - \beta) \bar{c}}{\alpha n} = q \bar{w} - \frac{\bar{w} \gamma}{\bar{r} + 1 - \delta} \quad (47)$$

$$\phi A_{10} \bar{k}_1^{\phi-1} \bar{l}_1^\mu N^{*\phi+\mu-1} = \theta A_{20} (\bar{k} - \bar{k}_1)^{\theta-1} (1 - \bar{l}_1 - qn)^{1-\theta} = \bar{r} \quad (48)$$

$$\mu A_{10} \bar{k}_1^\phi \bar{l}_1^{\mu-1} N^{\phi+\mu-1} = (1 - \theta) A_{20} (\bar{k} - \bar{k}_1)^\theta (1 - \bar{l}_1 - qn)^{-\theta} = \bar{w} \quad (49)$$

$$\bar{c} + \bar{k} \gamma n = A_{10} \bar{k}_1^\phi \bar{l}_1^\mu \bar{N}^{\phi+\mu-1} + A_{20} (\bar{k} - \bar{k}_1)^\theta (1 - \bar{l}_1 - qn)^{1-\theta} + (1 - \delta) \bar{k} \quad (50)$$

We next solve for the unknowns analytically. Equations (48) and (49) give

$$\frac{\phi \bar{l}_1}{\mu \bar{k}_1} = \frac{\theta (1 - \bar{l}_1 - qn)}{(1 - \theta) (\bar{k} - \bar{k}_1)},$$

so the system of steady state equations can be rewritten as

$$\begin{aligned} \frac{\gamma n}{\beta} - (1 - \delta) &= \bar{r} \\ \phi A_{10} \bar{k}_1^{\phi-1} \bar{l}_1^\mu \bar{N}^{\phi+\mu-1} &= \bar{r} \\ \phi (1 - \theta) \bar{l}_1 (\bar{k} - \bar{k}_1) &= \theta \mu (1 - \bar{l}_1 - qn) \bar{k}_1 \\ \frac{\alpha n \bar{w}}{(1 - \alpha - \beta)} \left(q - \frac{\gamma}{\bar{r} + 1 - \delta} \right) + \bar{k} \gamma n &= A_{10} \bar{k}_1^\phi \bar{l}_1^\mu \bar{N}^{\phi+\mu-1} + A_{20} (\bar{k} - \bar{k}_1)^\theta (1 - \bar{l}_1 - qn)^{1-\theta} + (1 - \delta) \bar{k} \\ (1 - \theta) A_{20} (\bar{k} - \bar{k}_1)^\theta (1 - \bar{l}_1 - qn)^{-\theta} &= \bar{w} \\ \theta A_{20} (\bar{k} - \bar{k}_1)^{\theta-1} (1 - \bar{l}_1 - qn)^{1-\theta} &= \bar{r} \end{aligned}$$

The first equation gives \bar{r} . From the third equation we have $\frac{(\bar{k} - \bar{k}_1)}{(1 - \bar{l}_1 - qn)} = \frac{\mu \theta}{\phi (1 - \theta)} \frac{\bar{k}_1}{\bar{l}_1}$ and from the last equation we get $\frac{(\bar{k} - \bar{k}_1)}{(1 - \bar{l}_1 - qn)} = \left(\frac{\bar{r}}{\theta A_{20}} \right)^{\frac{1}{\theta-1}}$. Hence, these two equations together imply that $\frac{\mu \theta}{\phi (1 - \theta)} \frac{\bar{k}_1}{\bar{l}_1} = \left(\frac{\bar{r}}{\theta A_{20}} \right)^{\frac{1}{\theta-1}}$ and hence,

$$\bar{l}_1 = \frac{\mu \theta}{\phi (1 - \theta)} \left(\frac{\bar{r}}{\theta A_{20}} \right)^{\frac{1}{1-\theta}} \bar{k}_1. \quad (51)$$

From the last equation we solve for $(1 - \bar{l}_1 - qn) = \left(\frac{\bar{r}}{\theta A_{20}} \right)^{\frac{1}{1-\theta}} (\bar{k} - \bar{k}_1)$. If we substitute for \bar{l}_1 in terms of \bar{k}_1 from equation (51) found above, we get

$$\begin{aligned} 1 - \frac{\mu \theta}{\phi (1 - \theta)} \left(\frac{\bar{r}}{\theta A_{20}} \right)^{\frac{1}{1-\theta}} \bar{k}_1 - qn &= \left(\frac{\bar{r}}{\theta A_{20}} \right)^{\frac{1}{1-\theta}} (\bar{k} - \bar{k}_1), \\ \bar{k}_1 &= \frac{\phi (1 - \theta)}{\mu \theta - \phi (1 - \theta)} \left(\frac{1 - qn}{\left(\frac{\bar{r}}{\theta A_{20}} \right)^{\frac{1}{1-\theta}}} - \bar{k} \right). \end{aligned}$$

Defining

$$\chi \equiv \left(\frac{\bar{r}}{\theta A_{20}} \right)^{\frac{1}{1-\theta}} \text{ and } \zeta \equiv \frac{\phi (1 - \theta)}{\mu \theta - \phi (1 - \theta)}$$

enables us to compactly rewrite

$$\bar{l}_1 = \frac{\mu \theta}{\phi (1 - \theta)} \chi \bar{k}_1, \quad (52)$$

$$\bar{k}_1 = \zeta \left(\frac{1 - qn}{\chi} - \bar{k} \right). \quad (53)$$

Substituting for $\bar{w} = \frac{\bar{r}\mu\bar{k}_1}{\phi\bar{l}_1}$, \bar{l}_1 and \bar{k}_1 in feasibility, we obtain a linear equation with one unknown \bar{k} ,

$$\begin{aligned} \frac{\alpha n \bar{w}}{(1-\alpha-\beta)} \left(q - \frac{\gamma}{\bar{r}+1-\delta} \right) + \bar{k} \gamma n &= A_{10} \bar{k}_1^\phi \bar{l}_1^\mu \bar{N}^{\phi+\mu-1} + A_{20} (\bar{k} - \bar{k}_1)^\theta (1 - \bar{l}_1 - qn)^{1-\theta} + (1-\delta)\bar{k}, \\ \frac{\alpha n}{(1-\alpha-\beta)} \left(q - \frac{\gamma}{\bar{r}+1-\delta} \right) \frac{\bar{r}\mu}{\phi \frac{\mu\theta}{\phi(1-\theta)} \chi} + \bar{k} \gamma n &= \frac{\bar{r}}{\phi} \zeta \left(\frac{1-qn}{\chi} - \bar{k} \right) + \frac{\bar{r} \left(\bar{k} - \zeta \left(\frac{1-qn}{\chi} - \bar{k} \right) \right)}{\theta} + (1-\delta)\bar{k}, \\ \bar{k} &= \left[\frac{\alpha n}{(1-\alpha-\beta)} \left(q - \frac{\gamma}{\bar{r}+1-\delta} \right) \frac{(1-\theta)\bar{r}}{\theta\chi} + \frac{(1-qn)\zeta\bar{r}}{\chi} \left(\frac{1}{\theta} - \frac{1}{\phi} \right) \right] / \left[\frac{(1+\zeta)\bar{r}}{\theta} - \frac{\bar{r}}{\phi} \zeta + 1 - \delta - \gamma n \right]. \end{aligned}$$

Hence, we solve for \bar{k} and then we can find \bar{k}_1 , \bar{l}_1 using (52) and (53) and finally use $\phi A_{10} \bar{k}_1^{\phi-1} \bar{l}_1^\mu \bar{N}^{\phi+\mu-1} = \bar{r}$ to find \bar{N} ,

$$\bar{N} = \left(\frac{\phi A_{10} \bar{k}_1^{\phi-1} \bar{l}_1^\mu}{\bar{r}} \right)^{\frac{1}{1-\phi-\mu}} \Lambda. \quad (54)$$

By construction of efficiency variables and given the fact that the original solution considered in this proposition exhibits Malthus-Solow behavior from period 0 and onward, we have that the steady state values of efficiency variables correspond to the values of original variables in time period 0. In particular, initial conditions (k_0, N_0) correspond to (\bar{k}, \bar{N}) . Furthermore, because the original solution satisfies certain restrictions, in particular, $r_0 > 0$, $k_0 > 0$, $0 < k_{10} < k_0$, $0 < l_{10} < 1 - qn_0$, $c_0 > 0$, $N_0 > 0$, it follows that (30) – (35) must hold. ■

Note that the above proposition implies that for a given set of parameter values, $\hat{\theta}$, there exists at most one pair (k_0, N_0) such that $\hat{\theta}$, $(k_0, N_0) \in S_1^$.*

Proposition 9 *If a given $\hat{\theta}$ satisfies (30) – (35), then $(k_0, N_0) = (\bar{k}, \bar{N})$ generates the solution exhibiting Malthus-Solow behavior from period 0 and onward.*

Proof. The assumption is that an admissible solution to the steady state values of efficiency variables exists. The claim is that $\hat{\theta}$, $(\bar{k}, \bar{N}) \in S_1^*$.

We start the economy at $(k_0, N_0) = (\bar{k}, \bar{N})$. Consider a candidate solution consisting of sequences $\{c_t = \bar{c}\gamma^t$, $N_t = \bar{N}n^t$, $k_t = \bar{k}\gamma^t$, $k_{1t} = \bar{k}_1$, $n_t = n\}_{t=0}^\infty$. We will show that this candidate solution satisfies all the sufficient equilibrium conditions (See Proposition 5) and exhibits the Malthus-Solow property. In light of definitions of efficiency variables, the proposed sequences satisfy equations (36) – (40) $\forall t$. This proposed solution is such that original variables grow at Malthus-Solow BGP growth rates (determined by (23) – (29)) and efficiency variables remain in the steady state. On this equilibrium path,

$$y_{1t}/y_t = \frac{A_{10} \bar{k}_1^\phi \bar{l}_1^\mu (\Lambda/\bar{N})^{1-\phi-\mu}}{A_{10} \bar{k}_1^\phi \bar{l}_1^\mu (\Lambda/\bar{N})^{1-\phi-\mu} + A_{20} (\bar{k} - \bar{k}_1)^\theta (1 - qn - \bar{l}_1)^{1-\theta}}$$

is constant $\forall t$, i.e. $\hat{\theta}$, $(k_0 = \bar{k}, N_0 = \bar{N}) \in S_1^*$.

Note that the candidate solution satisfies the assumptions of Proposition 5, hence, transversality conditions (8) and (9) also hold.

We must also check that the solution satisfies (13), i.e. that it is always optimal to operate,

$$\left(\frac{\phi A_{1t} K_t^{\phi-1} L_t^\mu \Lambda^{1-\phi-\mu}}{\theta} \right)^\theta \left(\frac{\mu A_1 K_t^\phi L_t^{\mu-1} \Lambda^{1-\phi-\mu}}{1-\theta} \right)^{1-\theta} \frac{1}{A_{2t}} \leq 1, \quad \forall t.$$

In what follows we show that this inequality is implied by (30) – (35).

The above inequality holds $\forall t$ if and only if it holds for $t = 0$ and the growth rate of $A_{2t} \geq$ the growth rate of $\left(\frac{\phi A_{1t} K_t^{\phi-1} L_t^\mu \Lambda^{1-\phi-\mu}}{\theta}\right)^\theta \left(\frac{\mu A_1 K_t^\phi L_t^{\mu-1} \Lambda^{1-\phi-\mu}}{1-\theta}\right)^{1-\theta}$; these two conditions are given by

$$\begin{aligned} [MS \text{ Ineq } 1] \quad &: \quad A_{20} \geq A_{10} \left(\frac{\phi}{\theta}\right)^\theta \left(\frac{\mu}{1-\theta}\right)^{1-\theta} \Lambda^{1-\phi-\mu} \bar{k}^{\phi-\theta} (1-qn)^{\mu+\theta-1} \bar{N}^{\phi+\mu-1}, \\ [MS \text{ Ineq } 2] \quad &: \quad n \geq \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}}\right)^{\frac{1}{1-\phi-\mu}}. \end{aligned}$$

Clearly, $[MS \text{ Ineq } 2]$ holds with equality because of (23) – (24).

Consider $[MS \text{ Ineq } 1]$.

$$\begin{aligned} A_{20} &\geq A_{10} \left(\frac{\phi}{\theta}\right)^\theta \left(\frac{\mu}{1-\theta}\right)^{1-\theta} \Lambda^{1-\phi-\mu} \bar{k}^{\phi-\theta} (1-qn)^{\mu+\theta-1} \bar{N}^{\phi+\mu-1}, \\ A_{20} \Lambda^{1-\phi-\mu} \frac{\phi A_{10} (\rho_k \bar{k})^{\phi-1} \bar{l}_1^\mu}{r} &\geq A_{10} \left(\frac{\phi}{\theta}\right)^\theta \left(\frac{\mu}{1-\theta}\right)^{1-\theta} \Lambda^{1-\phi-\mu} \bar{k}^{\phi-\theta} (1-qn)^{\mu+\theta-1}, \\ A_{20} \frac{\phi \rho_k^{\phi-1} \bar{l}_1^\mu}{r} &\geq \left(\frac{\phi}{\theta}\right)^\theta \left(\frac{\mu}{1-\theta}\right)^{1-\theta} \left(\left(\frac{r}{\theta A_{20}}\right)^{\frac{1}{\theta-1}} \frac{1-\bar{l}_1-qn}{1-\rho_k}\right)^{1-\theta} (1-qn)^{\mu+\theta-1}, \\ \rho_k^{\phi-1} \bar{l}_1^\mu &\geq \left(\frac{\theta}{1-\theta} \frac{\mu}{\phi}\right)^{1-\theta} \left(\frac{1-\bar{l}_1-qn}{1-\rho_k}\right)^{1-\theta} (1-qn)^{\mu+\theta-1}. \end{aligned}$$

Using (27) and (28), we have

$$\frac{1-l_1-qn}{1-\rho_k} \frac{\theta}{1-\theta} \frac{\mu}{\phi} = \frac{l_1}{\rho_k}, \quad (55)$$

so the above inequality reduces to

$$\begin{aligned} \rho_k^{\phi-1} l_1^\mu &\geq \left(\frac{l_1}{\rho_k}\right)^{1-\theta} (1-qn)^{\mu+\theta-1}, \\ \rho_k^{\phi-\theta} &\geq \left(\frac{l_1}{1-qn}\right)^{1-\theta-\mu}. \end{aligned} \quad (56)$$

Note that the following is true

$$\frac{k_1}{l_1} \leq \frac{k_2}{l_2}, \text{ i.e., } \frac{\rho_k}{\rho_l} \leq \frac{1-\rho_k}{1-\rho_l} \text{ iff } \rho_k \leq \frac{l_1}{1-qn} \equiv \rho_l. \quad (57)$$

Indeed, assuming $\rho_k \leq \rho_l$ we have $1-\rho_k > 1-\rho_l$ and hence $\frac{\rho_k}{\rho_l} \leq \frac{1-\rho_k}{1-\rho_l}$. Conversely, assuming $\frac{\rho_k}{\rho_l} \leq \frac{1-\rho_k}{1-\rho_l}$ and supposing $\rho_k > \rho_l$ gives $1-\rho_k < 1-\rho_l$, a contradiction.

Note that $\phi - \theta < 1 - \theta - \mu$ holds because we assume $0 < 1 - \mu - \phi$.

Case 1. If $\frac{\phi}{\mu} < \frac{\theta}{1-\theta}$ (it is then necessary that $\theta > \phi$) then (55) implies $\frac{k_1}{l_1} < \frac{k_2}{l_2}$ and hence $0 < \rho_k < \frac{l_1}{1-qn} < 1$ according to (57). Since $-1 < \phi - \theta < 1 - \theta - \mu < 1$, we have $\rho_k^{\phi-\theta} \geq \left(\frac{l_1}{1-qn}\right)^{1-\theta-\mu}$ and hence $[MS \text{ Ineq } 1]$ holds.

Case 2. If $\frac{\phi}{\mu} > \frac{\theta}{1-\theta}$ then (55) implies $\frac{k_1}{l_1} > \frac{k_2}{l_2}$ and hence $1 > \rho_k > \frac{l_1}{1-qn} > 0$ according to (57). Then we either have $\phi > \theta$ or $\phi < \theta$.

Case 2.a. $\phi > \theta$, and hence, $0 < \phi - \theta < 1 - \theta - \mu < 1$. Then (56) holds.

Case 2.b. $\phi < \theta$. Note then $-1 < \phi - \theta < 1 - \theta - \mu < 0$ cannot be the case. Indeed, assume $1 - \theta - \mu < 0$. Since $\frac{\phi}{\mu} > \frac{\theta}{1-\theta}$ we have $\phi - \phi\theta - \theta\mu > 0$, i.e. $\phi \left(1 - \theta - \frac{\theta}{\phi}\mu\right) > 0$ but $\frac{\theta}{\phi} > 1$ so we cannot have $1 - \theta - \mu < 0$.

It must be the case that $-1 < \phi - \theta < 0 < 1 - \theta - \mu < 1$. Then $\rho_k^{\phi-\theta} \geq \left(\frac{l_1}{1-qn}\right)^{1-\theta-\mu}$ holds.

Hence, [MS Ineq 1] holds. We showed that sufficient conditions for the competitive equilibrium are all satisfied and that the equilibrium paths exhibit MS BGP behavior from period 0 and onward. ■

Corollary 10 (to Propositions 8 and 9)

Part 1. Given an admissible solution $\gamma, n, r, l_1, \rho, \rho_k, \rho_y$ ($\gamma, n, r > 0, 0 < l_1 < 1 - qn, 1 - l_1 - qn > 0, 0 < \rho_k < 1$) to the system of equations (23) – (29), the following quantities,

$$\bar{r} = r, \quad \bar{l}_1 = l_1, \quad \bar{k} = \left(\frac{r}{\theta A_{20}}\right)^{\frac{1}{\theta-1}} \frac{1 - l_1 - qn}{1 - \rho_k}, \quad \bar{k}_1 = \rho_k \bar{k}, \quad \bar{c} = \rho \bar{k}, \quad \bar{N} = \left(\frac{\phi A_{10} \bar{k}_1^{\phi-1} \bar{l}_1^\mu}{r}\right)^{\frac{1}{1-\phi-\mu}} \Lambda,$$

satisfy (30) – (35).⁴

Part 2. Given admissible steady state efficiency variables $\bar{r}, \bar{c}, \bar{k}, \bar{k}_1, \bar{l}_1, \bar{N}$ (satisfying (30) – (35)) with γ and n determined by (23) and (24), the following quantities

$$\begin{aligned} \rho &= \frac{\bar{c}}{\bar{k}}, \quad \rho_k = \frac{\bar{k}_1}{\bar{k}}, \quad l_1 = \bar{l}_1, \quad \rho_y = \frac{A_{10} \bar{k}_1^\phi \bar{l}_1^\mu (\Lambda/\bar{N})^{1-\phi-\mu}}{A_{10} \bar{k}_1^\phi \bar{l}_1^\mu (\Lambda/\bar{N})^{1-\phi-\mu} + A_{20} (\bar{k} - \bar{k}_1)^\theta (1 - qn - \bar{l}_1)^{1-\theta}}, \\ r &= \theta A_{20} (\bar{k} - \bar{k}_1)^{\theta-1} (1 - qn - l_1)^{1-\theta} \end{aligned}$$

solve (25) – (29).⁵

Proof. It is straight-forward to verify this. ■

Proposition 11 (Part1) If $\hat{\theta} \in \Theta$ satisfies (30) – (35) then there exists at least 1 pair (k_0, N_0) such that $\hat{\theta}, (k_0, N_0) \in S_1$. (Part 2) If $\hat{\theta}, (k_0, N_0) \in S_1$ then $\hat{\theta} \in \Theta$ satisfies (30) – (35).

Before proving this proposition, we say a few words about the statement. Part 1 states that for a set of parameters satisfying (30) – (35), it is possible to find initial conditions such that equilibrium time paths exhibit Malthus-Solow BGP behavior in the limit. For most $\hat{\theta}$ satisfying (30) – (35) that we worked with, there exists a continuum of pairs (k_0, N_0) such that $\hat{\theta}, (k_0, N_0) \in S_1$. Identifying these, however, requires stability analysis, which cannot be performed analytically for this dynamical system. As discussed in the main text, when calibrating the model economy, we restrict our attention to S_1^* , i.e., we start the economy on a Malthus-Solow BGP. We then show numerically that this BGP is locally stable. This means that there is a continuum of initial conditions (k_0, N_0) that generate Malthus-Solow BGP behavior in the limit.

Part 2 of this proposition states that if equilibrium paths exhibit Malthus-Solow BGP behavior in the limit, then parameters must satisfy (30) – (35).

Proof.

(of Part 1) Implied by Proposition 9. The solution generated by $(k_0, N_0) = (\bar{k}, \bar{N})$ delivers Malthus-Solow BGP behavior from period 0 and onward, and hence, it gives Malthus-Solow BGP behavior in the limit.

(of Part 2) If $\hat{\theta}, (k_0, N_0) \in S_1$ then for t large enough, the economy approximately exhibits properties of a Malthus-Solow BGP, i.e. equations (23) – (29) have an admissible solution $\gamma, n, r, l_1, \rho, \rho_k, \rho_y$ (where $\rho = \frac{c}{k}$,

⁴These quantities represent steady state efficiency variables.

⁵These quantities characterize Malthus-Solow balanced growth path behavior.

$\rho_k = \frac{k_1}{\bar{k}}, \rho_y = \frac{y_1}{\bar{y}}$). This implies that the steady state levels of the corresponding efficiency variables (defined in the statement of Proposition 8) also have an admissible solution, backed out as follows (see Corollary 10),

$$\bar{r} = r, \quad \bar{l}_1 = l_1, \quad \bar{k} = \left(\frac{r}{\theta A_{20}} \right)^{\frac{1}{\theta-1}} \frac{1-l_1-qn}{1-\rho_k}, \quad \bar{k}_1 = \rho_k \bar{k}, \quad \bar{c} = \rho_k \bar{k}, \quad \bar{N} = \left(\frac{\phi A_{10} \bar{k}_1^{\phi-1} \bar{l}_1^\mu}{r} \right)^{\frac{1}{1-\phi-\mu}} \Lambda.$$

Since $r > 0$, $0 < l_1 < 1 - qn$, $1 - l_1 - qn > 0$, $0 < \rho_k < 1$, we have that $\hat{\theta} \in \Theta$ satisfies (30) – (35). ■

4.1.2 Existence of Malthus-Solow balanced growth

Here, we show that the Malthus-Solow (MS) BGP exists for a set of parameter values and initial conditions of positive measure. This is done in two steps.

(1) We first show that the Malthus-Solow BGP implied by the calibration is locally structurally stable. In other words, for sufficiently small changes in the calibrated parameter values, the limiting behavior of the solution is of Malthus-Solow type; i.e. it remains qualitatively unchanged. Starting with the calibrated parameter values, we perturb each parameter, to slightly larger and smaller values, while holding the rest of the parameters fixed. Table 1 reports the results for the original calibration and the results obtained with 18 different parameter calibrations (2 perturbations of each of the 9 parameters that enter the MS BGP equations). For each of these parameter combinations, a MS BGP exists. We show this by analytically finding an admissible solution to the system of the MS BGP equations 91-97. Note that the solution is admissible if

$$0 < \rho_k, \rho_y, 1 - l_1 - qn < 1.$$

The three figures illustrate that for each parameter combination there is a solution that is admissible; i.e. ρ_k, ρ_y and $1 - l_1 - qn$ fall within the appropriate range of $(0, 1)$.

Thus, we have effectively shown that there are no restrictions on the relationship between the parameter values needed to ensure the existence of an admissible solution to the MS BGP equations. Also note that because we find these solutions analytically and they are quantitatively different, precision here is not an issue; i.e., it is not the case that we are finding the original solution merely with less precision.

(2) Second, we show that each of the resulting MS BGPs (from Table 1) is locally Lyapunov stable. In other words, small changes in the initial conditions of the convergent paths do not cause a change in the limiting behavior of the equilibrium path. We show this by numerically log-linearizing the system of detrended variables and obtaining the eigenvalues (in Matlab): exactly two eigenvalues are less than 1 in each case, which is what is the necessary and sufficient condition for local stability of a dynamical system with two state variables (N_t and k_t). The eigenvalues are reported in Table 2.

We have also confirmed these results through extensive numerical testing: For small enough changes in the calibrated parameter values, the asymptotic path of the solution to the model remains qualitatively unchanged (both sectors operate forever), although of course the solution changes quantitatively. For small enough changes in the initial conditions k_0 and N_0 , the asymptotic solution remains quantitatively unchanged.

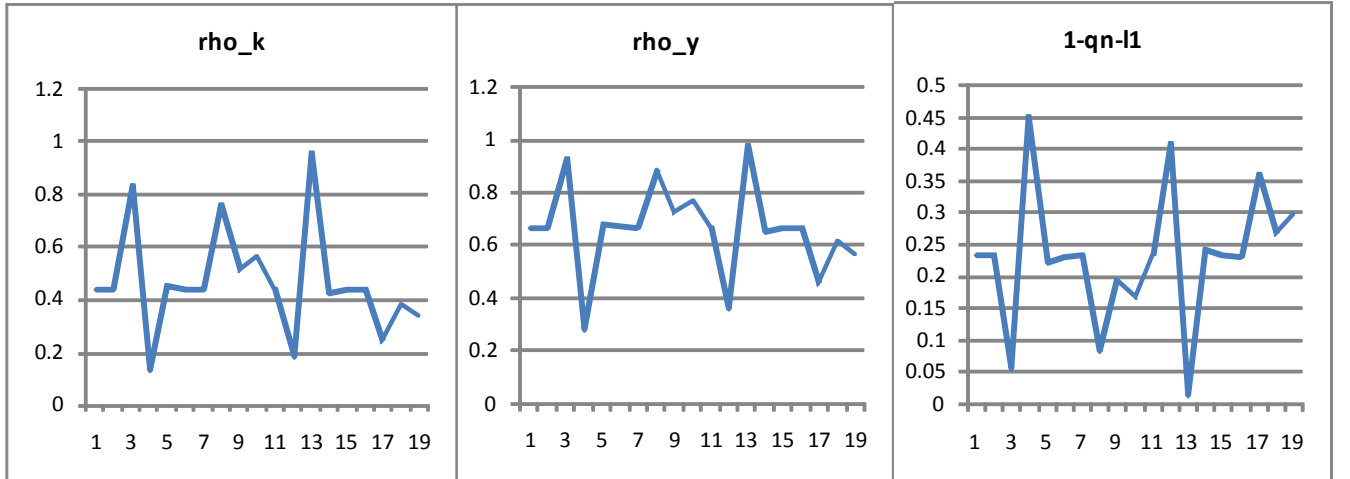


Table 1: The first row reports the result for our calibration. The other 18 rows report the results associated with perturbations of each parameter value, up and down, by .044 percent. For each parameter combination, the (analytical) solution exists and it is admissible (See Figures 1,2, and 3 below).

Parameters									Unknowns (Endogenous Moments of the Malthus-Solow BGP)						
δ	γ_1	γ_2	μ	ϕ	θ	α	β	q	γ	n	r	ρ_k	ρ	ρ_y	l_1
0.7226	1.0417	1.0062	0.5373	0.10448	0.2727	0.5818	0.4151	0.3828	1.0086	1.0972	2.3884	0.438	14.1	0.67	0.348
0.7229	1.0417	1.0062	0.5373	0.10448	0.2727	0.5818	0.4151	0.3828	1.0086	1.0972	2.3888	0.438	14.1	0.67	0.348
0.7226	1.0422	1.0062	0.5373	0.10448	0.2727	0.5818	0.4151	0.3828	1.0086	1.0986	2.3917	0.833	19.7	0.93	0.525
0.7226	1.0417	1.0067	0.5373	0.10448	0.2727	0.5818	0.4151	0.3828	1.0092	1.0956	2.386	0.131	9.76	0.28	0.13
0.7226	1.0417	1.0062	0.5375	0.10448	0.2727	0.5818	0.4151	0.3828	1.0086	1.0973	2.3886	0.454	14.33	0.68	0.357
0.7226	1.0417	1.0062	0.5373	0.10452	0.2727	0.5818	0.4151	0.3828	1.0086	1.0973	2.3885	0.441	14.14	0.67	0.35
0.7226	1.0417	1.0062	0.5373	0.10448	0.2728	0.5818	0.4151	0.3828	1.0086	1.0972	2.3884	0.436	14.08	0.67	0.347
0.7226	1.0417	1.0062	0.5373	0.10448	0.2727	0.582	0.4151	0.3828	1.0086	1.0972	2.3884	0.759	18.63	0.89	0.498
0.7226	1.0417	1.0062	0.5373	0.10448	0.2727	0.5818	0.4153	0.3828	1.0086	1.0972	2.3873	0.507	15.07	0.73	0.386
0.7226	1.0417	1.0062	0.5373	0.10448	0.2727	0.5818	0.4151	0.3829	1.0086	1.0972	2.3884	0.559	15.81	0.77	0.411
0.7223	1.0417	1.0062	0.5373	0.10448	0.2727	0.5818	0.4151	0.3828	1.0086	1.0972	2.3881	0.437	14.1	0.67	0.348
0.7226	1.0413	1.0062	0.5373	0.10448	0.2727	0.5818	0.4151	0.3828	1.0086	1.0959	2.3852	0.178	10.43	0.36	0.171
0.7226	1.0417	1.0058	0.5373	0.10448	0.2727	0.5818	0.4151	0.3828	1.0079	1.0989	2.3909	0.956	21.44	0.98	0.566
0.7226	1.0417	1.0062	0.5371	0.10448	0.2727	0.5818	0.4151	0.3828	1.0086	1.0972	2.3883	0.421	13.87	0.66	0.339
0.7226	1.0417	1.0062	0.5373	0.10443	0.2727	0.5818	0.4151	0.3828	1.0086	1.0972	2.3884	0.434	14.05	0.67	0.346
0.7226	1.0417	1.0062	0.5373	0.10448	0.2726	0.5818	0.4151	0.3828	1.0086	1.0972	2.3885	0.439	14.12	0.67	0.349
0.7226	1.0417	1.0062	0.5373	0.10448	0.2727	0.5815	0.4151	0.3828	1.0086	1.0972	2.3884	0.242	11.34	0.45	0.221
0.7226	1.0417	1.0062	0.5373	0.10448	0.2727	0.5818	0.4149	0.3828	1.0086	1.0972	2.3896	0.381	13.3	0.62	0.315
0.7226	1.0417	1.0062	0.5373	0.10448	0.2727	0.5818	0.4151	0.3826	1.0086	1.0972	2.3884	0.333	12.63	0.57	0.285

4.1.3 Comparative statics results for the Malthus-Solow BGP

Note that on a Malthus-Solow BGP, both, population growth and per capita output growth, are determined by the TFP growth rates in the two sectors,⁶

$$\gamma = \gamma_2^{\frac{1}{1-\theta}}, \quad n = \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}} \right)^{\frac{1}{1-\phi-\mu}}. \quad (58)$$

The growth rate of per capita output increases in the Solow TFP and is independent of the Malthusian TFP. Population growth increases in the Malthusian TFP growth rate and decreases in the Solow TFP growth rate. Interestingly, the time cost of raising children does not enter these two equations. This means that increasing the probability of survival while keeping all other parameters fixed would directly result in the proportional reduction of fertility ($n = \pi f$). For this class of simulations, i.e. in which we raise π and in which both the original and limiting behavior of equilibrium paths exhibits Malthus-Solow BGP properties, we found that during the transition from the original to the new balanced growth path, population growth exhibits a hump, and that this transition is lengthy. Therefore, it is misleading to conclude from these comparative statics exercises that mortality changes do not affect population growth.

It is important to notice that this analysis is only valid as long as the new value of π does not alter the type of limiting behavior of equilibrium paths, i.e., as long as it does not preclude convergence to a new Malthus-Solow BGP. In fact, in the simulation results of the benchmark economy that are presented in the main text, both of the exogenous changes (one is changes in γ_1 and γ_2 and one is changes in π) that are fed into the model imply that the economy converges to a Solow BGP.

⁶This result comes from the constancy of the interest rate on any balanced growth path and equality of the marginal products of capital in the two sectors. Hence, it is robust to the choice of the objective function.

Table 2: This table reports the eigenvalues for the log-linearized dynamical system of the detrended variables, parameterized according to the corresponding row of Table 1.

Eigenvalues of the Log-Linearized System of the dynamical system written in term of detrended variables					
Non-stable		Stable		Zero	
8.159	2.416	0.997	0.295	0	0
8.159	2.416	0.997	0.295	0	1E-15
8.155	2.421	0.995	0.295	0	1E-15
8.163	2.411	0.999	0.295	1E-15	1E-15
8.158	2.417	0.997	0.295	0	0
8.159	2.416	0.997	0.295	0	0
8.155	2.416	0.997	0.295	0	0
8.155	2.419	0.996	0.295	0	1E-15
8.154	2.416	0.997	0.295	1E-15	1E-15
8.157	2.418	0.996	0.295	1E-15	1E-15
8.158	2.416	0.997	0.295	0	0
8.162	2.412	0.999	0.295	0	0
8.153	2.422	0.994	0.295	0	1E-15
8.159	2.416	0.997	0.295	0	1E-15
8.159	2.416	0.997	0.295	0	0
8.162	2.416	0.997	0.295	0	1E-15
8.161	2.414	0.998	0.295	0	0
8.163	2.417	0.997	0.295	0	0
8.16	2.415	0.998	0.295	0	0

4.1.4 Derivation and discussion of Malthusian balanced growth properties

Next we derive the properties of the limiting behavior of equilibrium time paths in S_2 . Here we consider equilibrium paths along which all variables grow at constant rates, although not necessarily the same, and the Solow sector does not operate. The constancy of the growth rates allows us to simplify relevant equilibrium conditions and arrive at equilibrium growth rates of such a solution as functions of parameters.

Proposition 12 *If there is a solution $\{c_t, N_t, k_t, y_t\}_{t=0}^{\infty}$ such that all variables grow at constant rates $\forall t$, say $\gamma_c, n, \gamma_k, \gamma_y$, and the Solow sector never operates ($y_{2t} = 0 \forall t$), then the following is true.*

(1) $\gamma_c = \gamma_k = \gamma_y \equiv \gamma$.

(2) The unknowns γ, n, r, ρ (where $\rho = \frac{c}{k}$) are determined by the following system of equations,

$$\gamma_1 \gamma^{\phi-1} = n^{1-\phi-\mu}, \quad (59)$$

$$\gamma n = \beta(r + 1 - \delta), \quad (60)$$

$$\frac{(1 - \alpha - \beta) \rho \phi (1 - qn)}{\alpha n \mu r} = q - \frac{\gamma}{r + 1 - \delta}, \quad (61)$$

$$\rho + \gamma n = \frac{r}{\phi} + (1 - \delta). \quad (62)$$

(3) Corresponding efficiency variables, defined as follows,

$$c_t^* = \frac{c_t}{\gamma^t}, \quad k_t^* = \frac{k_t}{\gamma^t}, \quad N_t^* = \frac{N_t}{n^t},$$

with γ and n given by (59) – (62), are in steady state for all t (which we denote by a bar), $\bar{c}, \bar{k}, \bar{N} > 0$ and

satisfy (63) – (66), [M Ineq 1] and [M Ineq 2] given below,

$$\gamma_1 \gamma^{\phi-1} = n^{1-\phi-\mu}, \quad (63)$$

$$\gamma n = \beta \left[\phi A_{10} \bar{k}^{\phi-1} (1 - qn)^\mu \left(\frac{\Lambda}{\bar{N}} \right)^{1-\phi-\mu} + 1 - \delta \right], \quad (64)$$

$$\left(\frac{1 - \alpha - \beta}{\alpha} \right) \bar{c} = \mu A_{10} \bar{k}^\phi (1 - qn)^{\mu-1} \left(\frac{\Lambda}{\bar{N}} \right)^{1-\phi-\mu} (qn - \beta), \quad (65)$$

$$\bar{c} + \bar{k} \gamma n = A_{10} \bar{k}^\phi (1 - qn)^\mu \left(\frac{\Lambda}{\bar{N}} \right)^{1-\phi-\mu} + (1 - \delta) \bar{k}, \quad (66)$$

$$[M \text{ Ineq } 1] : 1 < \frac{A_{10}}{A_{20}} \left(\frac{\phi}{\theta} \right)^\theta \left(\frac{\mu}{1 - \theta} \right)^{1-\theta} \Lambda^{1-\phi-\mu} \bar{k}^{\phi-\theta} (1 - qn)^{\mu+\theta-1} \bar{N}^{\phi+\mu-1},$$

$$[M \text{ Ineq } 2] : n \leq \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}} \right)^{\frac{1}{1-\phi-\mu}}.$$

(4) Initial conditions (k_0, N_0) generating such a solution correspond to (\bar{k}, \bar{N}) .

Proof. The solution considered here (such that the Solow technology is never operated, i.e., $k_{1t} = k_t$ and $k_{2t} = k_t$, $\forall t$) must satisfy

$$\frac{c_{t+1}}{c_t} = \frac{\beta}{n_t} (r_{t+1} + 1 - \delta), \quad (67)$$

$$\frac{(1 - \alpha - \beta) c_t}{\alpha n_t} = q_t w_t - \frac{w_{t+1}}{r_{t+1} + 1 - \delta}, \quad (68)$$

$$c_t + k_{t+1} n_t = A_{10} \gamma_1^t k_t^\phi (1 - qn_t)^\mu \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu} + (1 - \delta) k_t, \quad (69)$$

$$r_t = \phi A_{10} \gamma_1^t k_t^{\phi-1} (1 - qn_t)^\mu \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu}, \quad (70)$$

$$w_t = \mu A_{10} \gamma_1^t k_t^\phi (1 - qn_t)^{\mu-1} \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu}, \quad (71)$$

where $n_t = \frac{N_{t+1}}{N_t}$ and the following inequality stating that the unit cost in the Solow sector is greater than 1, in other words, inequality (13) is reversed (see optimal resource allocation condition (14)),

$$\begin{aligned} 1 &< \frac{1}{A_{2t}} \left(\frac{\phi A_{1t} K_t^{\phi-1} L_t^\mu \Lambda^{1-\phi-\mu}}{\theta} \right)^\theta \left(\frac{\mu A_{1t} K_t^\phi L_t^{\mu-1} \Lambda^{1-\phi-\mu}}{1 - \theta} \right)^{1-\theta}, \text{ i.e.,} \\ 1 &< \frac{A_{1t}}{A_{2t}} \left(\frac{\phi}{\theta} \right)^\theta \left(\frac{\mu}{1 - \theta} \right)^{1-\theta} \Lambda^{1-\phi-\mu} K_t^{(\phi-1)\theta + \phi(1-\theta)} L_t^{\mu\theta + (\mu-1)(1-\theta)}. \end{aligned} \quad (72)$$

From the first equation, which becomes $\gamma_c = \frac{\beta}{n} [r_{t+1} + 1 - \delta]$ on a BGP, we see that r_t must remain constant, so we replace it by r . Then the constancy of r together with its definition in (70) imply that on such a BGP, $y_t = A_{10} \gamma_1^t k_t^\phi (1 - qn)^\mu \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu}$ grows at the same rate as k_t , so $\gamma_y = \gamma_k$.

Next we want to show that c_t and k_t must grow at the same constant rate. Consider (69) rewritten using the definition of r from (70),

$$\begin{aligned} c_t + k_{t+1} n_t &= \frac{r k_t}{\phi} + (1 - \delta) k_t, \\ \frac{c_t}{k_t} &= \frac{r}{\phi} + (1 - \delta) - \frac{k_{t+1}}{k_t} n. \end{aligned}$$

The right hand side is a constant, hence, the left hand side must also remain constant, denote it by $\rho = \frac{c_t}{k_t}$. So, c_t and k_t must grow at the same rate too. Hence, we have, $\gamma_c = \gamma_k = \gamma_y \equiv \gamma$. We use (70) together with the constancy of r once again to pin down the relationship between γ and n ,

$$\gamma_1 \gamma^{\phi-1} = n^{1-\phi-\mu}.$$

Output can be rewritten as

$$A_{1t} k_t^\phi (1 - qn)^\mu \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu} = \frac{r k_t}{\phi} \text{ or as } A_{1t} k_t^\phi (1 - qn)^\mu \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu} = \frac{w_t (1 - qn)}{\mu}.$$

This means that we can solve for w_t in terms of k_t : $w_t = \frac{\mu r k_t}{\phi(1-qn)}$. Then

$$\frac{c_t}{w_t} = \frac{c_t \phi (1 - qn)}{\mu r k_t} = \rho \frac{\phi (1 - qn)}{\mu r}.$$

Hence, we get a system of four equations (59) – (62) in four unknowns (γ, n, r, ρ) that describes properties of a Malthus BGP. So far, we proved results (1) and (2) of the proposition.

Rewriting (67) – (69) in terms of efficiency variables gives

$$\begin{aligned} \frac{c_{t+1}^* \gamma}{c_t^*} &= \frac{\beta N_t^*}{N_{t+1}^* n} \left[\phi A_{10} k_{t+1}^{*\phi-1} \left(1 - q_t \frac{N_{t+2}^* n}{N_{t+1}^*} \right)^\mu \left(\frac{\Lambda}{N_{t+1}^*} \right)^{1-\phi-\mu} + 1 - \delta \right], \\ \frac{(1 - \alpha - \beta) c_t^* N_t^*}{\alpha N_{t+1}^* n} &= q_t \mu A_{10} k_t^{*\phi} \left(1 - q_t \frac{N_{t+1}^* n}{N_t^*} \right)^{\mu-1} \left(\frac{\Lambda}{N_t^*} \right)^{1-\phi-\mu} - \\ &\quad - \frac{\mu A_{10} k_{t+1}^{*\phi} \gamma \left(1 - q_{t+1} \frac{N_{t+2}^* n}{N_{t+1}^*} \right)^{\mu-1} \left(\frac{\Lambda}{N_{t+1}^*} \right)^{1-\phi-\mu}}{\phi A_{10} k_{t+1}^{*\phi-1} \left(1 - q_t \frac{N_{t+2}^* n}{N_{t+1}^*} \right)^\mu \left(\frac{\Lambda}{N_{t+1}^*} \right)^{1-\phi-\mu} + 1 - \delta}, \\ c_t^* + k_{t+1}^* \frac{N_{t+1}^*}{N_t^*} \gamma n &= A_{10} k_t^{*\phi} \left(1 - q_t \frac{N_{t+1}^* n}{N_t^*} \right)^\mu \left(\frac{\Lambda}{N_t^*} \right)^{1-\phi-\mu} + (1 - \delta) k_t^*, \end{aligned}$$

Whenever the original variables are on a Malthus BGP, the efficiency variables are in steady state. This is true by construction of efficiency variables (that utilized information on γ and n along a Malthus BGP). Hence, the above system must hold when we replace efficiency variables by their constant steady state values, denoted by a bar. The above equations simplify to (64) – (66). Equation (63) holds because (59) holds (as already shown) on a Malthus BGP. We also have that $\bar{c}, \bar{k}, \bar{N}$ correspond to c_0, k_0 and N_0 , respectively. Since the original variables, c_0, k_0 and N_0 , are positive, we have also have $\bar{c}, \bar{k}, \bar{N} > 0$.

By optimal resource allocation condition (14), we have that inequality (72) must also hold $\forall t$. Substituting from the equilibrium condition, $L_t = (1 - qn) N_t$ and from $N_t = \bar{N} n^t$, $K_t = \bar{k} \bar{N} (\gamma n)^t$, $A_{1t} = A_{10} \gamma_1^t$, $A_{2t} = A_{20} \gamma_2^t$, we rewrite this inequality as

$$1 < \frac{A_{10} \gamma_1^t}{A_{20} \gamma_2^t} \left(\frac{\phi}{\theta} \right)^\theta \left(\frac{\mu}{1 - \theta} \right)^{1-\theta} \Lambda^{1-\phi-\mu} (\bar{k} \bar{N} (\gamma n)^t)^{(\phi-1)\theta+\phi(1-\theta)} ((1 - qn) \bar{N} n^t)^{\mu\theta+(\mu-1)(1-\theta)}. \quad (73)$$

For this inequality to hold for all t , it must hold for $t = 0$ and the growth rate of the denominator of the right hand side must not exceed the growth rate of the numerator. In other words, the following two conditions must hold,

$$\begin{aligned} 1 &< \frac{A_{10}}{A_{20}} \left(\frac{\phi}{\theta} \right)^\theta \left(\frac{\mu}{1 - \theta} \right)^{1-\theta} \Lambda^{1-\phi-\mu} (\bar{k} \bar{N})^{(\phi-1)\theta+\phi(1-\theta)} ((1 - qn) \bar{N})^{\mu\theta+(\mu-1)(1-\theta)}, \\ \gamma_2 &\leq \gamma_1 (\gamma n)^{(\phi-1)\theta+\phi(1-\theta)} n^{\mu\theta+(\mu-1)(1-\theta)}, \end{aligned}$$

which simplify to exactly [M Ineq 1] and [M Ineq 2]. ■

Proposition 13 *If $\hat{\theta} \in \Theta$ is such that for some arbitrary $\bar{N} > 0$ the system of equations (63) – (66) has a solution $\gamma, n, \bar{c}, \bar{k}$ such that $\gamma, n, \bar{c}, \bar{k} > 0$, [M Ineq 1] and [M Ineq 2] are satisfied, then $(k_0, N_0) = (\bar{k}, \bar{N})$ generates the solution exhibiting Malthusian balanced growth behavior from period 0 and onward.*

Proof. This proposition assumes that $\hat{\theta}$ is such that for some arbitrary \bar{N} , an admissible solution to the steady state values of efficiency variables that correspond to a Malthus BGP exists and both [M Ineq 1] and [M Ineq 2] hold. We want to show that $\hat{\theta}, (\bar{k}, \bar{N}) \in S_2^*$.

We start the economy off at $k_0 = \bar{k}$ and $N_0 = \bar{N}$. Consider sequences $\{k_t = \bar{k}\gamma^t, c_t = \bar{c}\gamma^t, N_t = \bar{N}n^t\}_{t=0}^\infty$ as a candidate solution. We will show that this proposed solution satisfies all sufficient conditions for an equilibrium exhibiting Malthus BGP properties. This solution satisfies equations (67) – (71) $\forall t$. Since the assumptions of Proposition 5 hold, the transversality conditions (8) and (9) also hold. Because [M Ineq 1] and [M Ineq 2] are satisfied, we have that inequality (73) holds $\forall t$. Substituting $N_t = \bar{N}n^t, K_t = \bar{k}\bar{N}(\gamma n)^t, A_{1t} = A_{10}\gamma_1^t, A_{2t} = A_{20}\gamma_2^t$ into this inequality gives (72) $\forall t$, that is, the Solow technology is never used. Hence, the proposed solution, $\{k_t = \bar{k}\gamma^t, c_t = \bar{c}\gamma^t, N_t = \bar{N}n^t\}_{t=0}^\infty$, satisfies conditions sufficient to be the equilibrium solution. Hence, $\hat{\theta}, (k_0 = \bar{k}, N_0 = \bar{N}) \in S_2^*$. ■

Corollary 14 *(to Propositions 12 and 13)*

Suppose there is a solution $\{c_t, N_t, k_t, y_t\}_{t=0}^\infty$ such that all variables grow at constant rates $\forall t$ and the Solow sector never operates ($y_{2t} = 0 \forall t$). Then given the solution γ, n, r, ρ to (59) – (62), we have that $\gamma,$

$n, \bar{N} = N_0, \bar{k} = \left(\frac{r}{\phi A_{10}(1-qn)^\mu \left(\frac{\Lambda}{N}\right)^{1-\phi-\mu}} \right)^{\frac{1}{\phi-1}}$ and $\bar{c} = \rho\bar{k}$ solve (63) – (66) and satisfy [M Ineq 1] and [M Ineq 2].⁷ Conversely, given a solution $\gamma, n, \bar{N}, \bar{k}, \bar{c}$ to (63) – (66) such that [M Ineq 1] and [M Ineq 2] are satisfied, we have $\gamma, n, \rho = \frac{\bar{c}}{\bar{k}}, r = \phi A_{10}\bar{k}^{\phi-1} (1-qn)^\mu \left(\frac{\Lambda}{N}\right)^{1-\phi-\mu}$ solve (59) – (62).

Proof. Straight-forward to verify. ■

Proposition 15 *(Part 1) If $\hat{\theta} \in \Theta$ is such that for some arbitrary $\bar{N} > 0$ the following system of equations (63) – (66) has a solution $\gamma, n, \bar{N}, \bar{c}, \bar{k}$ such that $\bar{c}, \bar{k} > 0$, [M Ineq 1] and [M Ineq 2] are satisfied, then there exists at least one k_0 such that $\hat{\theta}, (k_0, N_0 = \bar{N}) \in S_2$. (Part 2) If $\hat{\theta}, (k_0, N_0) \in S_2$, then the system of equations (63) – (66) has a solution $\gamma, n, \bar{c}, \bar{k}, \bar{N}$ such that $\bar{c}, \bar{k} > 0$ and [M Ineq 1] and [M Ineq 2] both hold.*

Proof.

(of Part 1) Part 1 of this proposition states that for a set of parameters and an arbitrary \bar{N} such that there is a solution to (63) – (66) for which [M Ineq 1] and [M Ineq 2] both hold, there exists at least one pair of initial conditions such that equilibrium time paths exhibit Malthusian balanced growth behavior in the limit. The proof is implied by Proposition 13. The solution generated by $(k_0, N_0) = (\bar{k}, \bar{N})$ exhibits Malthusian behavior from period 0 and onward, and hence, it also delivers Malthusian behavior in the limit.

Again, for most such $\hat{\theta}$, there is a continuum of k_0 such that $\hat{\theta}, (k_0, N_0 = \bar{N}) \in S_2$, but the restrictions on such $\hat{\theta}$ and k_0 are impossible to derive analytically.

(of Part 2) If $\hat{\theta}, (k_0, N_0) \in S_2$ then for t large enough, say $t > t^*$, the economy approximately exhibits the properties of a Malthus BGP, i.e. equations (59) – (62) have an admissible solution γ, n, r, ρ and [M Ineq 2] holds either with = or with <. Indeed, if [M Ineq 2] did not hold, for t large enough, Solow sector would operate. Now pick any arbitrary $\bar{t} > t^*$ and let $\bar{N} = N_{\bar{t}}/n^{\bar{t}}$. Note that \bar{N} is unaffected by the choice of $\bar{t} > t^*$.

Then it is easy to verify that $\gamma, n, \bar{N}, \bar{k} = \left(\frac{r}{\phi A_{10}(1-qn)^\mu \left(\frac{\Lambda}{N}\right)^{1-\phi-\mu}} \right)^{\frac{1}{\phi-1}}, \bar{c} = \rho\bar{k}$ solve (63) – (66) and satisfy [M Ineq 1] and [M Ineq 2]. It is easily verified. (See Corollary 14). ■

⁷Notice \bar{N} influences \bar{k} and \bar{c} as well as [M Ineq 1].

Proposition 16 For a given $\hat{\theta}$, there may exist multiple pairs (k_0, N_0) such that $\hat{\theta}, (k_0, N_0) \in S_2^*$.

Proof. This is true because the system (63) – (66) together with [M Ineq 1] and [M Ineq 2] can have more than one solution. Consider the following θ as an example: $\alpha = 0.4, \beta = 0.2, \phi = 0.03, \mu = 0.71147, \Lambda = 1, \theta = 0.486, \delta = 1, A_1 = 100, A_2 = 100, a = 0.451, b = 0.259, \pi = 0.6706, \gamma_1 = 1.3845, \gamma_2 = 1.1165$. Then $(k_0, N_0) = (1.549, 0.00202)$ and $(k_0, N_0) = (1.86925, 0.001)$ both belong to S_2^* . ■

4.1.5 Comparative Statics Results for the Malthus BGP

- First we show that $\boxed{\frac{\partial n}{\partial \gamma_1} < 0}$ ($=0$ when $\delta = 1$).

Indeed, we can rewrite equations (59) – (62) as

$$\begin{aligned} \gamma_1 \gamma^{\phi-1} &= n^{1-\phi-\mu}, \\ \frac{(1-\alpha-\beta)\phi(1-qn)}{\alpha\mu\left(\frac{\gamma n}{\beta} - (1-\delta)\right)} \left(\frac{\frac{\gamma n}{\beta} - (1-\delta)}{\phi} + (1-\delta) - \gamma n \right) &= qn - \beta, \end{aligned}$$

where $r = \frac{\gamma n}{\beta} - (1-\delta)$ and $\rho = \frac{r}{\phi} + (1-\delta) - \gamma n$.

Let $\kappa = \frac{(1-\alpha-\beta)}{\alpha}$,

$$\begin{aligned} \gamma_1 \gamma^{\phi-1} &= n^{1-\phi-\mu}, \\ (1-qn) \left(\frac{1}{\phi} - \frac{\gamma n - (1-\delta)}{\left(\frac{\gamma n}{\beta} - (1-\delta)\right)} \right) &= \frac{\mu}{\kappa\phi} (qn - \beta), \end{aligned}$$

The first equation gives $\left(\frac{\gamma_1}{n^{1-\phi-\mu}}\right)^{\frac{1}{1-\phi}} = \gamma$ and hence $\gamma n = \gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}$. Substituting into the second equation gives

$$(1-qn) \left(\frac{1}{\phi} - \frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta)}{\left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta)\right)} \right) = \frac{\mu}{\kappa\phi} (qn - \beta). \quad (74)$$

The above equation implicitly defines n . Note that if $\delta = 1$, n is independent of γ , so in that case $\frac{\partial n}{\partial \gamma_1} = 0$. Totally differentiating w.r.t. γ_1 obtains

$$\begin{aligned} -q \frac{\partial n}{\partial \gamma_1} \left(\frac{1}{\phi} - \frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta)}{\left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta)\right)} \right) &- (1-qn) \frac{1}{1-\phi} \left\{ \left(\gamma_1^{\frac{\phi}{1-\phi}} n^{\frac{\mu}{1-\phi}} + \mu \gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu-1+\phi}{1-\phi}} \right) \left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta) \right) \right. \\ &\left. - \frac{1}{\beta} \left(\gamma_1^{\frac{\phi}{1-\phi}} n^{\frac{\mu}{1-\phi}} + \mu \gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu-1+\phi}{1-\phi}} \right) \left(\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta) \right) \right\} / \left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta) \right)^2 = \frac{\mu}{\kappa\phi} q \frac{\partial n}{\partial \gamma_1}, \text{ i.e.} \\ -q \frac{\partial n}{\partial \gamma_1} \left(\frac{1}{\phi} - \frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta)}{\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta)} \right) &- (1-qn) \frac{1}{\beta} \frac{1}{1-\phi} \left(\gamma_1^{\frac{\phi}{1-\phi}} n^{\frac{\mu}{1-\phi}} + \mu \gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu-1+\phi}{1-\phi}} \right) \frac{(1-\beta)(1-\delta)}{\left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta) \right)^2} = \\ &= \frac{\mu}{\kappa\phi} q \frac{\partial n}{\partial \gamma_1}. \end{aligned}$$

Note that $\frac{1}{\phi} - \frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta)}{\left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta)\right)} > 0$. Indeed, $r = \frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta) > 0$ and $\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta)$ could

have either sign. If it is negative then we are done. However if it is positive then $\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta)}{\left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta)\right)} < 1$,

so subtracting something less than 1 from $\frac{1}{\phi}$ yields a positive value. Hence, we have

$$-q \frac{\partial n}{\partial \gamma_1} (+) - (1 - qn) (+) = \frac{\mu}{\kappa \phi} q \frac{\partial n}{\partial \gamma_1}$$

Note that $1 - qn > 0$ because this is labor supply per adult. It is clear from the above equation that $\frac{\partial n}{\partial \gamma_1} < 0$. Otherwise, we would have a negative LHS but a positive RHS.

- Since $\gamma = \left(\frac{\gamma_1}{n^{1-\phi-\mu}}\right)^{\frac{1}{1-\phi}}$ and we already showed that $\frac{\partial n}{\partial \gamma_1} < 0$, we have $\boxed{\frac{\partial \gamma}{\partial \gamma_1} > 0}$.
- We also have $\frac{\partial n}{\partial q} < 0$ (equivalently, $\frac{\partial n}{\partial \pi} > 0$).

Consider (74) again,

$$\begin{aligned} (1 - qn) \left(\frac{1}{\phi} - \frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta)}{\left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta)\right)} \right) &= \frac{\mu}{\kappa \phi} (qn - \beta). \\ (1 - qn) \left(\frac{1}{\theta} - \frac{\gamma n - (1-\delta)}{\frac{\gamma n}{\beta} - (1-\delta)} \right) &= \frac{(1-\theta)}{\kappa \theta} (qn - \beta) \end{aligned}$$

Recall that the second term of the product on the LHS is positive. This means that both $(1 - qn)$ and $(qn - \beta)$ are positive. (Indeed, both of these expressions being negative would imply $1 < qn < \beta$, a contradiction). Suppose contrary to the claim, we have $\frac{\partial n}{\partial q} > 0$. Then as q increases, n also increases. Then unambiguously, $(1 - qn)$ declines and $(qn - \beta)$ increases. For equation (74) above to hold, $\frac{1}{\phi} - \frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta)}{\left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta)\right)}$

must increase. This leads to a contradiction. Indeed,

$$\begin{aligned} \frac{d}{dn} \left(\frac{1}{\phi} - \frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta)}{\left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta)\right)} \right) &= \\ - \frac{\frac{\mu}{1-\phi} \gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu-1+\phi}{1-\phi}} \left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta) \right) - \frac{1}{\beta} \frac{\mu}{1-\phi} \gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu-1+\phi}{1-\phi}} \left(\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta) \right)}{\left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta) \right)^2} &= \\ = - \frac{\left(\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - \beta(1-\delta) \right) - \left(\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}} - (1-\delta) \right)}{\left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta) \right)^2} &= - \frac{(1-\delta)(1-\beta)}{\left(\frac{\gamma_1^{\frac{1}{1-\phi}} n^{\frac{\mu}{1-\phi}}}{\beta} - (1-\delta) \right)^2} < 0. \end{aligned}$$

- Finally, since we have $\gamma = \left(\frac{\gamma_1}{n^{1-\phi-\mu}}\right)^{\frac{1}{1-\phi}}$ and we already showed that $\frac{\partial n}{\partial q} < 0$, we also have $\frac{\partial \gamma}{\partial q} > 0$. (As survival probability increases, q declines, population growth rate rises, dampening economic growth.

4.1.6 Derivation and discussion of Solow balanced growth properties

Recall from Lemma 2 that it is always optimal to operate the Malthusian technology. This means that a Solow balanced growth path can only emerge asymptotically, with the Malthusian output relative to the total converging to zero, as defined in S_3 . In order to have a discussion and derive equations describing Solow balanced growth, we eliminate the Malthusian technology from existence by setting $A_{1t} = 0$ for all t . In such a system, there is only one state variable, k_t . Initial condition N_0 only determines population dynamics. Optimal per capita variables are independent of $\{N_t\}_{t=0}^\infty$.

Proposition 17 *Consider a version of our model with $A_{1t} = 0, \forall t$. If there is a solution $\{c_t, N_t, k_t, y_t\}_{t=0}^\infty$ such that all variables grow at constant rates, say $\gamma_c, n, \gamma_k, \gamma_y$ then the following is true.*

- (1) $\gamma_c = \gamma_k = \gamma_y \equiv \gamma$.
(2) The unknowns γ, n, r, ρ (where $\rho = \frac{c}{k}$) are determined by the following system of equations,

$$\gamma = \gamma_2^{\frac{1}{1-\theta}} \quad (75)$$

$$\gamma n = \beta(r + 1 - \delta) \quad (76)$$

$$\frac{(1 - \alpha - \beta)}{\alpha} \rho \frac{\theta(1 - qn)}{(1 - \theta)r} = qn - \beta \quad (77)$$

$$\rho + \gamma n = \frac{r}{\theta} + (1 - \delta) \quad (78)$$

- (3) Corresponding efficiency variables, defined as follows,

$$c_t^* = \frac{c_t}{\gamma^t}, \quad k_t^* = \frac{k_t}{\gamma^t}, \quad N_t^* = \frac{N_t}{n^t},$$

with γ and n given by (75) – (78), are in steady state for all t (which we denote by a bar), $\bar{c}, \bar{k} > 0$ and satisfy (79) – (82), [S Ineq 1] and [S Ineq 2] given below,

$$\gamma = \gamma_2^{\frac{1}{1-\theta}} \quad (79)$$

$$\gamma n = \beta \left[\theta A_{20} \bar{k}^{\theta-1} (1 - qn)^{1-\theta} + 1 - \delta \right] \quad (80)$$

$$\left(\frac{1 - \alpha - \beta}{\alpha} \right) \bar{c} = (1 - \theta) A_{20} \bar{k}^\theta (1 - qn)^{-\theta} (qn - \beta) \quad (81)$$

$$\bar{c} + \bar{k} \gamma n = A_{20} \bar{k}^\theta (1 - qn)^{1-\theta} + (1 - \delta) \bar{k} \quad (82)$$

- (4) Initial condition k_0 generating such a solution correspond to \bar{k} .

Proof. The proposed solution must allocate zero resources to the Malthusian sector, $y_{1t} = 0$ (because $A_{1t} = 0$). Hence, conditions that the equilibrium solution must satisfy (with $A_{1t} = 0$) are given by

$$\frac{c_{t+1}}{c_t} = \frac{\beta}{n_t} (r_{t+1} + 1 - \delta), \quad (83)$$

$$\frac{(1 - \alpha - \beta) c_t}{\alpha n_t} = q_t w_t - \frac{w_{t+1}}{r_{t+1} + 1 - \delta}, \quad (84)$$

$$c_t + k_{t+1} n_t = A_{20} \gamma_2^t k_t^\theta (1 - q_t n_t)^{1-\theta} + (1 - \delta) k_t, \quad (85)$$

$$r_t = \theta A_{20} \gamma_2^t k_t^{\theta-1} (1 - q_t n_t)^{1-\theta}, \quad (86)$$

$$w_t = (1 - \theta) A_{20} \gamma_2^t k_t^\theta (1 - q_t n_t)^{-\theta}. \quad (87)$$

From the first equation, which becomes $\gamma_c = \frac{\beta}{n} [r_{t+1} + 1 - \delta]$ on a BGP, we see that r_t must remain constant, so we replace it by r . Then the constancy of r together with its definition in (86) imply that

$$\gamma_k = \gamma_2^{\frac{1}{1-\theta}} \quad (88)$$

Since output can be rewritten as $A_t k_t^\theta (1 - qn)^{1-\theta} = \frac{\theta}{\theta} A_t k_t^{\theta-1} k_t (1 - qn)^{1-\theta} = \frac{rk_t}{\theta}$, we have $\gamma_y = \gamma_k$. Next we want to show that c_t and k_t must grow at the same constant rate. Consider (85) rewritten using the definition of r ,

$$\begin{aligned} c_t + k_{t+1}n_t &= \frac{rk_t}{\phi} + (1 - \delta)k_t, \\ \frac{c_t}{k_t} &= \frac{r}{\phi} + (1 - \delta) - \frac{k_{t+1}}{k_t}n. \end{aligned}$$

The right hand side is a constant, hence, the left hand side must also remain constant, denote it by $\rho = \frac{c_t}{k_t}$. So, c_t and k_t must grow at the same rate too. Hence, using (88) we have

$$\gamma_c = \gamma_k = \gamma_y \equiv \gamma = \gamma_2^{\frac{1}{1-\theta}}. \quad (89)$$

Further, from the second equation (84) it is seen that c_t and w_t must grow at the same rate. ■
Note that output can also be rewritten as either $\frac{rk_t}{\theta}$ or

$$A_t k_t^\theta (1 - qn)^{1-\theta} = \frac{1 - \theta}{1 - \theta} A_t k_t^\theta (1 - qn)^{-\theta} (1 - qn) = \frac{w_t (1 - qn)}{1 - \theta},$$

i.e., we can solve for w_t in terms of k_t . We obtain $w_t = \frac{(1-\theta)rk_t}{\theta(1-qn)}$. Then we obtain

$$\frac{c_t}{w_t} = \frac{c_t \theta (1 - qn)}{k_t (1 - \theta) r} = \rho \frac{\theta (1 - qn)}{(1 - \theta) r},$$

which mean that (84) can be written as

$$\frac{(1 - \alpha - \beta)}{\alpha} \rho \frac{\theta (1 - qn)}{(1 - \theta) r} = qn - \beta.$$

Hence, we obtain a system of four equations (75) – (78) in four unknowns γ, n, r, ρ . So far we proved parts 1 and 2 of the proposition.

Rewriting (83) – (85) in terms of efficiency variables gives

$$\begin{aligned} \frac{c_{t+1}^* \gamma}{c_t^*} &= \frac{\beta N_t^*}{N_{t+1}^* n} \left[\theta A_{20} k_{t+1}^{*\theta-1} \left(1 - q_t \frac{N_{t+2}^* n}{N_{t+1}^*} \right)^{1-\theta} + 1 - \delta \right], \\ \frac{(1 - \alpha - \beta)}{\alpha} \frac{c_t^* N_t^*}{N_{t+1}^* n} &= q_t (1 - \theta) A_{20} k_t^{*\theta} \left(1 - q_t \frac{N_{t+1}^* n}{N_t^*} \right)^{-\theta} - \frac{\gamma (1 - \theta) A_{20} k_{t+1}^{*\theta} \left(1 - q_{t+1} \frac{N_{t+2}^* n}{N_{t+1}^*} \right)^{-\theta}}{\theta A_{20} k_{t+1}^{*\theta-1} \left(1 - q_{t+1} \frac{N_{t+2}^* n}{N_{t+1}^*} \right)^{1-\theta} + 1 - \delta}, \\ c_t^* + k_{t+1}^* \frac{N_{t+1}^*}{N_t^*} \gamma n &= A_{20} k_t^{*\theta} \left(1 - q_t \frac{N_{t+1}^* n}{N_t^*} \right)^{1-\theta} + (1 - \delta) k_t^*. \end{aligned}$$

Whenever the original variables are on a Solow BGP, the efficiency variables are in steady state. This is true by construction of efficiency variables that used information on γ and n determined in (75) – (78). Hence, the above system must hold when we replace the efficiency variables by their constant steady state values, denoted by a bar. The above equations then simplify to (80) – (82). Equation (79) holds because we showed that equation (75) must hold. We also have that \bar{c}, \bar{k} correspond to c_0, k_0 , respectively. Because the original variables, c_0 and k_0 , are positive, we also have $\bar{c}, \bar{k} > 0$.

Proposition 18 *Consider a version of our model with $A_{1t} = 0, \forall t$. If $\hat{\theta} \in \Theta$ is such that the system of equations (79) – (82) has a solution $\gamma, n, \bar{c}, \bar{k}$ such that $\gamma, n, \bar{c}, \bar{k} > 0$, then (k_0, N_0) where $k_0 = \bar{k}$ and N_0 is any positive real number generates the solution exhibiting Solow balanced growth behavior from period 0 and onward.*

Proof. This proposition assumes that $\hat{\theta}$ is such that an admissible solution to the steady state values of efficiency variables that correspond to a Solow BGP (in an economy without the Malthusian technology) exists. We start the economy off at $k_0 = \bar{k}$ and N_0 being any positive real number. Consider sequences $\{k_t = \bar{k}\gamma^t, c_t = \bar{c}\gamma^t, N_t = N_0 n^t\}_{t=0}^\infty$ as a candidate solution. This solution satisfies equations (83) – (87) $\forall t$. Since the assumptions of Proposition 5 hold, the transversality conditions (8) and (9) also hold. Hence, the proposed solution satisfies conditions sufficient to be the equilibrium solution, and it exhibits Solow balanced growth properties. ■

Corollary 19 (to Propositions 17 and 18)

Consider a version of our model with $A_{1t} = 0, \forall t$. Suppose there is a solution $\{c_t, N_t, k_t, y_t\}_{t=0}^\infty$ such that all variables grow at constant rates $\forall t$. Then given the solution γ, n, r, ρ to (75) – (78), we have that $\gamma, n, \bar{k} = \left(\frac{r}{\theta A_{20}(1-qn)^{1-\theta}}\right)^{\frac{1}{\theta-1}}$ and $\bar{c} = \rho \bar{k}$ solve (79) – (82). Conversely, given a solution $\gamma, n, \bar{k}, \bar{c}$ to (79) – (82), we have $\gamma, n, \rho = \frac{\bar{c}}{\bar{k}}$ and $r = \theta A_{20} \bar{k}^{\theta-1} (1 - qn)^{1-\theta}$ solve (75) – (78).

Proof. Straight-forward to verify. ■

Proposition 20 If $\hat{\theta}, (k_0, N_0) \in S_3$ then the system (79) – (82) has a solution $\gamma, n, \bar{c}, \bar{k}$ such that $\bar{c}, \bar{k} = k_0 > 0$.

Proof. If $\hat{\theta}, (k_0, N_0) \in S_3$ then for t large enough the economy approximately exhibits the properties of a Solow, with the Malthusian sector output relative to the total converging to 0. Equations (75) – (78) then have an admissible solution γ, n, r, ρ . Then the steady state levels of efficiency variables have an admissible solution, as we can back these out as in Corollary 19, i.e., $\hat{\theta} \in \Theta$ satisfies (79) – (82). ■

4.1.7 Comparative Statics Results for the Solow BGP

Consider a version of our model with $A_{1t} = 0, \forall t$.

- First, it is clear from (75) that $\boxed{\frac{\partial \gamma}{\partial \gamma_2} > 0}$.
- It is also true that $\boxed{\frac{\partial n}{\partial \gamma_2} < 0}$ ($=0$ when $\delta = 1$).

Indeed, we can rewrite equations (75) – (78) as

$$\begin{aligned} \gamma &= \gamma_2^{\frac{1}{1-\theta}}, \\ \frac{(1-\alpha-\beta)}{\alpha} \frac{\theta(1-qn)}{(1-\theta)\left(\frac{\gamma n}{\beta} - (1-\delta)\right)} \left(\frac{\frac{\gamma n}{\beta} - (1-\delta)}{\theta} + (1-\delta) - \gamma n\right) &= qn - \beta, \end{aligned}$$

i.e. (let $\kappa = \frac{(1-\alpha-\beta)}{\alpha}$),

$$\begin{aligned} \gamma &= \gamma_2^{\frac{1}{1-\theta}}, \\ (1-qn) \left(\frac{1}{\theta} - \frac{\gamma n - (1-\delta)}{\frac{\gamma n}{\beta} - (1-\delta)}\right) &= \frac{(1-\theta)}{\kappa \theta} (qn - \beta). \end{aligned} \tag{90}$$

Note that if $\delta = 1$, the above equation defines n independently of γ , so in that case $\frac{\partial n}{\partial \gamma_2} = 0$.

Totally differentiating w.r.t. γ (which is a function of an exogenously given parameter γ_2) obtains

$$\begin{aligned} -q \frac{\partial n}{\partial \gamma} \left(\frac{1}{\theta} - \frac{\gamma n - (1 - \delta)}{\frac{\gamma n}{\beta} - (1 - \delta)} \right) + (1 - qn) \left(-\frac{n + \gamma \frac{\partial n}{\partial \gamma}}{\frac{\gamma n}{\beta} - (1 - \delta)} - \frac{(\gamma n - (1 - \delta))(-1)}{\left(\frac{\gamma n}{\beta} - (1 - \delta)\right)^2} \frac{1}{\beta} \left(n + \gamma \frac{\partial n}{\partial \gamma} \right) \right) &= \frac{(1 - \theta)}{\kappa \theta} q \frac{\partial n}{\partial \gamma}, \\ -q \frac{\partial n}{\partial \gamma} \left(\frac{1}{\theta} - \frac{\gamma n - (1 - \delta)}{\frac{\gamma n}{\beta} - (1 - \delta)} \right) + (1 - qn) \left(\frac{\gamma n - (1 - \delta)}{\left(\frac{\gamma n}{\beta} - (1 - \delta)\right)^2} \frac{1}{\beta} \left(n + \gamma \frac{\partial n}{\partial \gamma} \right) - \frac{n + \gamma \frac{\partial n}{\partial \gamma}}{\frac{\gamma n}{\beta} - (1 - \delta)} \right) &= \frac{(1 - \theta)}{\kappa \theta} q \frac{\partial n}{\partial \gamma}. \end{aligned}$$

Substituting for $\zeta \equiv \frac{\gamma n - (1 - \delta)}{\frac{\gamma n}{\beta} - (1 - \delta)}$ gives

$$\begin{aligned} -q \frac{\partial n}{\partial \gamma} \left(\frac{1}{\theta} - \zeta \right) + (1 - qn) \frac{\left(n + \gamma \frac{\partial n}{\partial \gamma} \right)}{\left(\frac{\gamma n}{\beta} - (1 - \delta) \right)} \left(\frac{\zeta}{\beta} - 1 \right) &= \frac{(1 - \theta)}{\kappa \theta} q \frac{\partial n}{\partial \gamma} \\ -q \left(\underbrace{\frac{1}{\theta} - \zeta}_+ \right) + (1 - qn) \frac{\left(n / \frac{\partial n}{\partial \gamma} + \gamma \right)}{\left(\frac{\gamma n}{\beta} - (1 - \delta) \right)} \left(\underbrace{\frac{\zeta}{\beta} - 1}_- \right) &= \underbrace{\frac{(1 - \theta)}{\kappa \theta} q}_+ \end{aligned}$$

Note that $r = \frac{\gamma n}{\beta} - (1 - \delta) > 0$, but ζ can be negative or positive. If it is positive then $\zeta < \beta$ because $\frac{\frac{\gamma n}{\beta} - (1 - \delta)}{\frac{\gamma n}{\beta} - (1 - \delta)} < 1$, so we have $\frac{1}{\theta} - \zeta > 0$ and $\frac{\zeta}{\beta} - 1 < 0$ in the above. It is then clear that $\frac{\partial n}{\partial \gamma}$ must be negative for the LHS to yield a positive number.

Since $\frac{\partial \gamma}{\partial \gamma_2} > 0$, we have $\frac{\partial n}{\partial \gamma_2} < 0$.

- Finally, we also have $\frac{\partial n}{\partial q} < 0$ (equivalently, $\frac{\partial n}{\partial \pi} > 0$)

Consider (90) again,

$$(1 - qn) \left(\frac{1}{\theta} - \frac{\gamma n - (1 - \delta)}{\frac{\gamma n}{\beta} - (1 - \delta)} \right) = \frac{(1 - \theta)}{\kappa \theta} (qn - \beta)$$

Recall that the second term of the product on the LHS is positive. This means that both $(1 - qn)$ and $(qn - \beta)$ are positive. (Indeed, both of these expressions being negative would imply $1 < qn < \beta$, a contradiction). Suppose contrary to the claim, we have $\frac{\partial n}{\partial q} > 0$. Then as q increases, n also increases. Then unambiguously, $(1 - qn)$ declines and $(qn - \beta)$ increases. For equation (90) above to hold, $\left(\frac{1}{\theta} - \frac{\gamma n - (1 - \delta)}{\frac{\gamma n}{\beta} - (1 - \delta)} \right)$ must increase. This leads to a contradiction. Indeed,

$$\begin{aligned} \frac{d}{dn} \left(\frac{1}{\theta} - \frac{\gamma n - (1 - \delta)}{\frac{\gamma n}{\beta} - (1 - \delta)} \right) &= -\frac{\gamma \left(\frac{\gamma n}{\beta} - (1 - \delta) \right) - \frac{\gamma}{\beta} (\gamma n - (1 - \delta))}{+} = \\ &= -\frac{(\gamma n - (1 - \delta)) \beta - (\gamma n - (1 - \delta))}{+} = -\frac{(1 - \delta)(1 - \beta)}{+} < 0. \end{aligned}$$

4.1.8 Summary of Balanced Growth Paths Properties and Comparative Statics Results

As discussed in the paper, equilibrium time paths may exhibit one of three possible types of limiting behavior. It is both the parameter values and initial conditions that determine which type of behavior the equilibrium paths will exhibit. It is instructive to present the equations determining the properties along each possible type of balanced growth. See Bar and Leukhina (2007) for derivations, propositions and proofs.

- (1) *Malthus-Solow balanced growth*, $\frac{y_{1t}(\theta, k_0, N_0)}{y_t(\theta, k_0, N_0)} = \rho_y \in (0, 1) \forall t$.

All per capita variables grow at the same rate, $\gamma_c = \gamma_k = \gamma_{k1} = \gamma_y = \gamma_{y1} \equiv \gamma$.

The unknowns $\gamma, n, r, l_1, \rho, \rho_k, \rho_y$ (where $\rho = \frac{c}{k}, \rho_k = \frac{k_1}{k}, \rho_y = \frac{y_1}{y}$) satisfy the following equations,⁸

$$\begin{aligned}\gamma &= \gamma_2^{\frac{1}{1-\theta}}, \\ n &= \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}} \right)^{\frac{1}{1-\phi-\mu}}, \\ \gamma &= \frac{\beta}{n} [r + 1 - \delta], \\ \frac{(1-\alpha-\beta) \rho \phi l_1}{\alpha n \mu r \rho_k} &= q - \frac{\gamma}{(r+1-\delta)}, \\ \frac{\theta \rho_k}{(1-\rho_k)} &= \frac{\phi \rho_y}{(1-\rho_y)}, \\ \frac{\mu \rho_y}{(1-\rho_y)} &= \frac{(1-\theta) l_1}{(1-l_1-qn)}, \\ \rho + \gamma n &= \frac{r \rho_k}{\rho_y \phi} + (1-\delta).\end{aligned}$$

Comparative statics results: $\frac{\partial n}{\partial \gamma_1} > 0$, $\frac{\partial n}{\partial \gamma_2} < 0$, $\frac{\partial \gamma}{\partial \gamma_1} = 0$, $\frac{\partial \gamma}{\partial \gamma_2} > 0$, $\frac{\partial n}{\partial q} = \frac{\partial \gamma}{\partial q} = 0$.

(2) *Malthus balanced growth*, $y_{2t}(\hat{\theta}, k_0, N_0) = 0 \forall t$.

All per capita variables grow at the same rate, $\gamma_c = \gamma_k = \gamma_y \equiv \gamma$. The unknowns γ, n, r, ρ (where $\rho = \frac{c}{k}$) are determined by the following system of equations,

$$\begin{aligned}\gamma_1 \gamma^{\phi-1} &= n^{1-\phi-\mu}, \\ \gamma n &= \beta (r + 1 - \delta), \\ \frac{(1-\alpha-\beta) \rho \phi (1-qn)}{\alpha n \mu r} &= q - \frac{\gamma}{r+1-\delta}, \\ \rho + \gamma n &= \frac{r}{\phi} + (1-\delta).\end{aligned}$$

A necessary condition for such balanced growth is that $n \leq \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}} \right)^{\frac{1}{1-\phi-\mu}}$, which ensures that employing Solow technology is never optimal.

Comparative statics results: $\frac{\partial n}{\partial \gamma_1} < 0$ ($=0$ if $\delta = 1$), $\frac{\partial \gamma}{\partial \gamma_1} > 0$, $\frac{\partial n}{\partial q} < 0$ (equivalently, $\frac{\partial n}{\partial \pi} > 0$), $\frac{\partial \gamma}{\partial q} > 0$.

(3) *Solow balanced growth*, $\frac{y_{1t}(\hat{\theta}, k_0, N_0)}{y_t(\hat{\theta}, k_0, N_0)} \sim 0$. Equations are derived under the assumption that $A_{1t} = 0 \forall t$.

All per capita variables grow at the same rate, $\gamma_c = \gamma_k = \gamma_y \equiv \gamma$. The unknowns γ, n, r, ρ (where $\rho = \frac{c}{k}$) are determined by the following system of equations,

$$\begin{aligned}\gamma &= \gamma_2^{\frac{1}{1-\theta}} \\ \gamma n &= \beta (r + 1 - \delta) \\ \frac{(1-\alpha-\beta) \theta (1-qn)}{\alpha} \rho \frac{(1-qn)}{(1-\theta) r} &= qn - \beta \\ \rho + \gamma n &= \frac{r}{\theta} + (1-\delta)\end{aligned}$$

Comparative statics results: $\frac{\partial n}{\partial \gamma_2} < 0$ ($=0$ if $\delta = 1$), $\frac{\partial \gamma}{\partial \gamma_2} > 0$, $\frac{\partial n}{\partial q} < 0$ (equivalently, $\frac{\partial n}{\partial \pi} > 0$), $\frac{\partial \gamma}{\partial q} = 0$.

4.1.9 Segmentation of the parameter and initial conditions space

Proposition 21 $\exists \hat{\theta} \in \Theta$ such that S_1, S_2 , and S_3 are all non-empty.

⁸There is a unique analytical solution to this system of equations, which is derived in Bar and Leukhina (2007).

Proof. It suffices to present an example. Consider the same $\hat{\theta}$ as in the proof of Proposition 16. Then $(k_0, N_0) = (1.549, 0.00202) \in S_2$, $(k_0, N_0) = (27.1564, 0.995) \in S_3$, and $(k_0, N_0) = (15.1665, 0.0044) \in S_1$. The initial condition then determines the limiting behavior of the equilibrium paths. For each such $\hat{\theta}$ that we found, S_1 coincides with S_1^* , that is, there is a unique initial condition that allows the equilibrium time paths to exhibit Malthus-Solow BGP behavior. Refer to Figure 2. The discussion here corresponds to the segment of $\hat{\theta}$ for which S_3 is depicted as a smaller part. For this segment S_1 consists of a line. ■

Figure 2 roughly illustrates how the space of parameter and initial condition values may be split into $S_1, S_1^*, S_2, S_2^*, S_3$ and other segments generating behavior omitted from the above discussion.

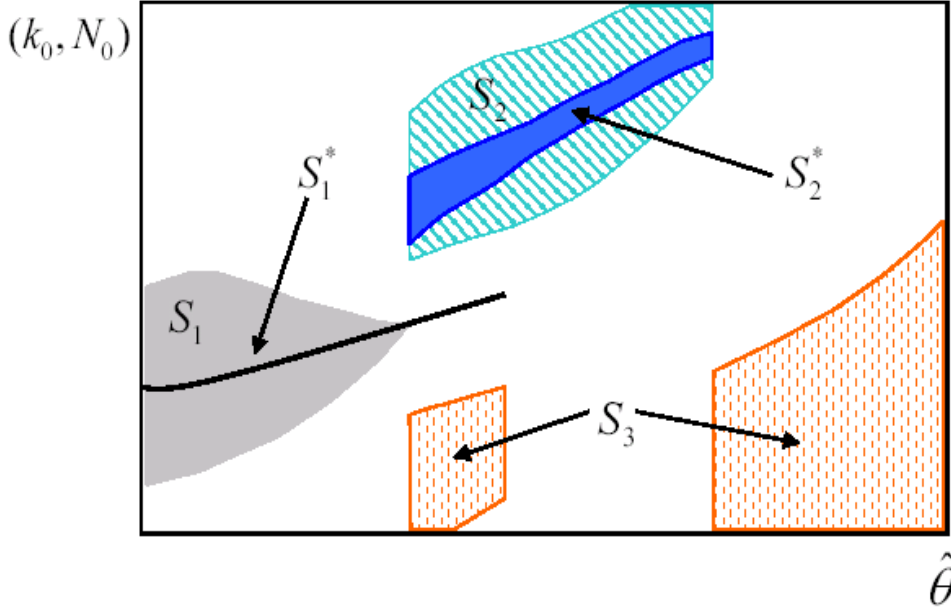


Figure 2. Stylized Segmentation of the Parameter and Initial Conditions Space

There is no analytical solution for the limiting growth rate of population and per capita output (γ and n) in S_2 and S_3 . The systems of equations determining γ and n in S_2 and S_3 are given by (59)–(62) and (75)–(78), respectively. The comparative statics results show that for both, Malthus BGP and Solow BGP, increases in the TFP growth rate lead to a decline in the population growth rate and an increase in per capita output growth rate. For Malthus BGPs, increases in probability of survival lead to exactly the opposite effect. In contrast, for Solow BGPs, increases in survival probabilities lead to increases in population growth but do not affect the growth rate of per capita output, $\gamma = \gamma_2^{\frac{1}{1-\theta}}$.

5 Calibration as a solution to the system of linear equations

We calibrate the model under the assumption that the English economy around 1600 is on a Malthus-Solow BGP. The system of equations that describes the properties of a Malthus-Solow BGP is given by

$$\gamma = \gamma_2^{\frac{1}{1-\theta}}, \quad (91)$$

$$n = \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}} \right)^{\frac{1}{1-\phi-\mu}}, \quad (92)$$

$$\gamma = \frac{\beta}{n} [r + 1 - \delta], \quad (93)$$

$$\frac{(1-\alpha-\beta)}{\alpha n} \frac{\rho \phi l_1}{\mu r \rho_k} = q - \frac{\gamma}{(r+1-\delta)}, \quad (94)$$

$$\frac{\theta \rho_k}{(1-\rho_k)} = \frac{\phi \rho_y}{(1-\rho_y)}, \quad (95)$$

$$\frac{\mu \rho_y}{(1-\rho_y)} = \frac{(1-\theta) l_1}{(1-l_1-qn)}, \quad (96)$$

$$\rho + \gamma n = \frac{r \rho_k}{\rho_y \phi} + (1-\delta), \quad (97)$$

where the unknowns are γ , n , r , l_1 , ρ , ρ_k , ρ_y . ($\rho = \frac{c}{k}$, $\rho_k = \frac{k_1}{k}$, $\rho_y = \frac{y_1}{y}$).

The idea is to rewrite the system of equations that describes the properties of a Malthus-Solow BGP (91)–(97) in terms of the available data moments $(\frac{l_1}{l}, \frac{y_1}{y}, \frac{wl}{y}, \frac{rk}{y})$ and parameters only.

First, labor share is available and hence it directly pins down μ : $\frac{wl}{y} = \frac{\mu y_1}{l_1} \frac{l}{y} \Rightarrow \mu$. Equation six is $w_t = \frac{\mu y_{1t}}{l_1} = \frac{(1-\theta)y_{2t}}{(1-l_1-qn)}$. Combine it with an algebraic identity to get

$$\begin{aligned} 1 - qn &= l_1 + 1 - l_1 - qn \\ wl &= wl_1 + w(1 - l_1 - qn) \\ \frac{wl}{y} &= \frac{\mu y_1}{l_1} \frac{l_1}{y} + \frac{(1-\theta)y_2}{(1-l_1-qn)} \frac{(1-l_1-qn)}{y} \\ \frac{wl}{y} &= \frac{\mu y_1}{y} + \frac{(1-\theta)y_2}{y} \end{aligned}$$

This pins down θ . Next, equation five is just $r \equiv \frac{\phi y_1}{\rho_k k} = \frac{\theta y_2}{(1-\rho_k)k}$. Combining it with an algebraic identity we get

$$\begin{aligned} k &= \rho_k k + (1-\rho_k)k \\ rk &= r \rho_k k + r(1-\rho_k)k \\ rk &= \frac{\phi y_1}{\rho_k k} \rho_k k + \frac{\theta y_2}{(1-\rho_k)k} (1-\rho_k)k \\ rk &= \phi y_1 + \theta y_2 \\ \frac{rk}{y} &= \frac{\phi y_1}{y} + \frac{\theta y_2}{y} \end{aligned}$$

This allows us to get γ_1 and γ_2 and give prediction to γ and n . Then $\gamma = \frac{\beta}{n} [r + 1 - \delta]$ can be used together with $r + 1 - \delta$ in the data to get β . We then use the moment qn to get $a/\pi + b$. Separately employing the assumption on a/b and π we calibrate a and b . Finally, we combine equations four and seven

$$\begin{aligned} \frac{(1-\alpha-\beta)}{\alpha n} \frac{\rho \phi l_1}{\mu r \rho_k} &= q - \frac{\gamma}{(r+1-\delta)} \\ \rho + \gamma n &= \frac{y}{k} + (1-\delta) \end{aligned}$$

to get

$$\begin{aligned}\frac{(1-\alpha-\beta)}{\alpha\mu} \frac{\phi l_1}{r\rho_k} \left(r \frac{y}{rk} + (1-\delta) - \gamma n\right) &= qn - \frac{\gamma n}{(r+1-\delta)} \\ \frac{(1-\alpha-\beta)(1-qn)}{\alpha\mu} \frac{\phi}{r\rho_k} \frac{l_1}{(1-qn)} \left(r \frac{y}{rk} + (1-\delta) - \gamma n\right) &= qn - \frac{\gamma n}{(r+1-\delta)}\end{aligned}$$

Since

$$\begin{aligned}r &= \frac{\phi y_1}{k_1} \\ \frac{k}{y_1} &= \frac{\phi}{r\rho_k}\end{aligned}$$

and hence, the above becomes

$$\frac{(1-\alpha-\beta)(1-qn)}{\alpha\mu} \frac{y}{y_1} \frac{1}{r} \frac{rk}{y} \frac{l_1}{(1-qn)} \left(r \frac{y}{rk} + (1-\delta) - \gamma n\right) = qn - \frac{\gamma n}{(r+1-\delta)}$$

which pins down α .

We can think of solving the last equation as solving 2 equations in two unknowns: ρ and α .

Hence, we solve the following equations

$$\frac{wl}{y} = \frac{\mu y_1}{l_1} \frac{l}{y} \quad (98)$$

$$\frac{wl}{y} = \frac{\mu y_1}{y} + \frac{(1-\theta)y_2}{y} \quad (99)$$

$$\frac{rk}{y} = \frac{\phi y_1}{y} + \frac{\theta y_2}{y} \quad (100)$$

$$\gamma = \gamma_2^{\frac{1}{1-\theta}} \quad (101)$$

$$n = \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}} \right)^{\frac{1}{1-\phi-\mu}} \quad (102)$$

$$\gamma = \frac{\beta}{n} [r+1-\delta] \quad (103)$$

$$\frac{qn}{n} = \frac{b}{\pi} \frac{a}{b} + b \quad (104)$$

$$a = b \frac{a}{b} \quad (105)$$

$$\frac{(1-\alpha-\beta)(1-qn)}{\alpha\mu} \frac{y}{y_1} \frac{1}{r} \frac{rk}{y} \frac{l_1}{(1-qn)} \rho = qn - \frac{\gamma n}{(r+1-\delta)} \quad (106)$$

$$\rho + \gamma n = r \frac{y}{rk} + (1-\delta) \quad (107)$$

Hence calibration can be summarized as a solution to a system of linear equations. In the system of linear equations above, $\pi, \delta, \gamma_1, \gamma_2$ are directly pinned down in the data, although γ_1 and γ_2 are pinned down only ϕ, μ, θ are determined. The system of equations consists of 10 equations in terms of 10 unknowns, 7 of which are parameters, $\mu, \phi, \theta, \beta, a, b, \alpha$, and 3 of which are moments that we do not take from the data: $\frac{c}{k}, \gamma, n$. Moments used are $\frac{wl}{y}, \frac{rk}{y}, \frac{y_1}{y}, \frac{l_1}{l}, qn, \frac{a}{b}, r$.

6 Solution method

The equilibrium time paths are sequences of allocations and prices that satisfy

$$\frac{C_{t+1}}{C_t} = \beta (r_{t+1} + 1 - \delta), \quad (108)$$

$$\frac{(1 - \alpha - \beta) C_{t+1}}{N_{t+1}} = \alpha \beta [(r_{t+1} + 1 - \delta) q_t w_t - w_{t+1}], \quad (109)$$

$$C_t + K_{t+1} = F(K_t, L_t; t) + (1 - \delta) K_t, \quad (110)$$

$$L_t = N_t - q_t N_{t+1}, \quad (111)$$

where

$$\begin{aligned} F(K_t, L_t; t) &\equiv \max_{K_{1t}, L_{1t}} \left[A_{1t} K_{1t}^{-\phi} L_{1t}^{\mu} \Lambda^{1-\phi-\mu} + A_{2t} (K_t - K_{1t})^{\theta} (L_t - L_{1t})^{1-\theta} \right] \\ \text{s.t. } 0 &\leq K_{1t} \leq K_t, \quad 0 \leq L_{1t} \leq L_t \end{aligned} \quad (112)$$

and

$$r_t = F_1(K_t, L_t; t), \quad w_t = F_2(K_t, L_t; t). \quad (113)$$

Notice that the time cost of raising a surviving child q_t as well as $A_{1t} = A_{10} \prod_{\tau=0}^t \gamma_{1\tau}$ and $A_{2t} = A_{20} \prod_{\tau=0}^t \gamma_{2\tau}$ are indexed by t . The experiments that we perform involve changing $\{q_t, \gamma_{1t}, \gamma_{2t}\}_{t=1600}^{t=2000}$ in accordance with historical data.

Conditions (108) – (113) rewritten in per household terms become

$$\frac{c_{t+1}}{c_t} = \beta \frac{N_t}{N_{t+1}} (r_{t+1} + 1 - \delta), \quad (114)$$

$$(1 - \alpha - \beta) c_{t+1} = \alpha \beta [(r_{t+1} + 1 - \delta) q_t w_t - w_{t+1}], \quad (115)$$

$$c_t + k_{t+1} \frac{N_{t+1}}{N_t} = f(k_t, l_t) + (1 - \delta) k_t, \quad (116)$$

$$l_t = 1 - q_n t, \quad (117)$$

where

$$\begin{aligned} f(k_t, l_t) &= \max_{k_{1t}, l_{1t}} \left\{ A_{1t} k_{1t}^{-\phi} l_{1t}^{\mu} \left(\frac{\Lambda}{N_t} \right)^{1-\phi-\mu} + A_{2t} (k_t - k_{1t})^{\theta} (l_t - l_{1t})^{1-\theta} \right\} \\ \text{s.t. } 0 &\leq k_{1t} \leq k_t, \quad 0 \leq l_{1t} \leq l_t, \end{aligned} \quad (118)$$

and

$$r_t = f_1(k_t, l_t), \quad w_t = f_2(k_t, l_t). \quad (119)$$

Since the equilibrium time paths exhibit exponential growth, it is difficult to directly search for the numerical solution that satisfies the above conditions. As is commonly done in practice, we work with efficiency, or detrended, variables defined as follows:

$$\begin{aligned} c_t^* &= \frac{c_t}{\prod_{\tau=0}^{t-1} g_{\tau}}, \quad k_t^* = \frac{k_t}{\prod_{\tau=0}^{t-1} g_{\tau}}, \quad k_{1t}^* = \frac{k_{1t}}{\prod_{\tau=0}^{t-1} g_{\tau}}, \quad N_t^* = \frac{N_t}{\prod_{\tau=0}^{t-1} g_{n\tau}}, \\ l_t^* &= l_t, \quad l_{1t}^* = l_{1t}, \quad w_t^* = \frac{w_t}{\prod_{\tau=0}^{t-1} g_{\tau}}, \quad r_t = r_t^*, \end{aligned} \quad (120)$$

where we assume that $\prod_{\tau=0}^{-1} g_{\tau} = \prod_{\tau=0}^{-1} g_{n\tau} = 1$ and g_t and g_{nt} represent the balanced growth rates of y_t and N_t respectively that correspond to the parameters at time t . For the discussion of determining the balanced growth path growth rates for a given parameter choice see the previous section on balanced growth. The reason why we use products of growth rates to detrend the original variables instead of powers of the original growth rate is again the fact that changing parameters might (and actually does) lead to a change of the

limiting growth rates. Hence, detrending the original variables by powers of the growth rates along the initial balanced growth path will not be sufficient to eliminate exponential growth of the unknown time paths.

We rewrite conditions (114) – (119) in terms of efficiency variables to obtain

$$\frac{c_{t+1}^* g_{nt}}{c_t^*} = \beta \frac{N_t^*}{N_{t+1}^*} (r_{t+1}^* + 1 - \delta), \quad (121)$$

$$(1 - \alpha - \beta) c_{t+1}^* g_t = \alpha \beta [(r_{t+1}^* + 1 - \delta) q_t w_t^* - w_{t+1}^* g_t], \quad (122)$$

$$c_t^* + k_{t+1}^* \frac{N_{t+1}^*}{N_t^*} g_{nt} = f(k_t^*, l_t^*) + (1 - \delta) k_t^*, \quad (123)$$

$$l_t^* = 1 - q_t \frac{N_{t+1}^* g_{nt}}{N_t^*}, \quad (124)$$

where

$$f(k_t^*, l_t^*) = \max_{k_{1t}^*, l_{1t}^*} \left\{ \tilde{A}_{1t} k_{1t}^{*\phi} l_{1t}^{*\mu} \tilde{\Lambda}_t^{1-\phi-\mu} + \tilde{A}_{2t} (k_t^* - k_{1t}^*)^\theta (l_t^* - l_{1t}^*)^{1-\theta} \right\} \quad (125)$$

$$\text{s.t. } 0 \leq k_{1t}^* \leq k_t^*, \quad 0 \leq l_{1t}^* \leq l_t^*, \quad \text{where}$$

$$\tilde{\Lambda}_t = \frac{\Lambda}{N_t^*},$$

$$\tilde{A}_{1t} = A_{10} \left(\prod_{\tau=0}^{t-1} \gamma_{1\tau} \right) \left(\prod_{\tau=0}^{t-1} g_\tau \right)^{\phi-1} \left(\prod_{\tau=0}^{t-1} g_{n\tau} \right)^{\mu+\phi-1},$$

$$\tilde{A}_{2t} = A_{20} \left(\prod_{\tau=0}^{t-1} \gamma_{2\tau} \right) \left(\prod_{\tau=0}^{t-1} g_\tau \right)^{\theta-1}$$

and

$$r_t^* = f_1(k_t^*, l_t^*), \quad w_t^* = f_2(k_t^*, l_t^*). \quad (126)$$

Hence, we search for equilibrium time paths of efficiency variables that satisfy conditions (121) – (126) using the original steady state efficiency variables as the initial guess. Once the equilibrium efficiency variables are obtained, we use (120) to back out the equilibrium time paths of the original variables.

7 Solving our model with the Barro and Becker parental utility

Proposition 22 *Under the assumption of $U_t = c_t^\sigma + \beta n_t^{1-\varepsilon} U_{t+1}$ (the Barro-Becker formulation), the objective function in (DP) can be replaced by $\sum_{t=0}^{\infty} \beta^t c_t^\sigma N_t^{1-\varepsilon}$.*

Proof.

$$\begin{aligned} U_0 &= c_0^\sigma + \beta n_0^{1-\varepsilon} U_1 = \\ &= c_0^\sigma + \beta n_0^{1-\varepsilon} (c_1^\sigma + \beta n_1^{1-\varepsilon} U_2) = \\ &= c_0^\sigma + \beta n_0^{1-\varepsilon} (c_1^\sigma + \beta n_1^{1-\varepsilon} (c_2^\sigma + \beta n_2^{1-\varepsilon} U_3)) = \dots \\ &= c_0^\sigma + \beta n_0^{1-\varepsilon} c_1^\sigma + \beta n_0^{1-\varepsilon} \beta n_1^{1-\varepsilon} c_2^\sigma + \beta n_0^{1-\varepsilon} \beta n_1^{1-\varepsilon} \beta n_2^{1-\varepsilon} U_3 = \dots \\ &= c_0^\sigma + \beta c_1^\sigma \left(\frac{N_1}{N_0} \right)^{1-\varepsilon} + \beta^2 c_2^\sigma \left(\frac{N_2}{N_0} \right)^{1-\varepsilon} + \beta^3 \left(\frac{N_3}{N_0} \right)^{1-\varepsilon} U_3 \\ &= \left(\frac{1}{N_0} \right)^{1-\varepsilon} \left(\sum_{t=0}^{\infty} \beta^t c_t^\sigma N_t^{1-\varepsilon} + \lim_{t \rightarrow \infty} \beta^t N_t^{1-\varepsilon} U_t \right) = \left(\frac{1}{N_0} \right)^{1-\varepsilon} \sum_{t=0}^{\infty} \beta^t c_t^\sigma N_t^{1-\varepsilon} \end{aligned}$$

Since N_0 is just a constant, the utility function can be replaced by $\sum_{t=0}^{\infty} \beta^t c_t^\sigma N_t^{1-\varepsilon}$. ■

Just like for the case of $U_t = \alpha \log c_t + (1 - \alpha) \log n_t + U_{t+1}$, the competitive equilibrium allocation for the case of the Barro-Becker utility can be found by solving the corresponding sequential problem,

$$\begin{aligned}
& \max_{\{C_t, N_{t+1}, K_{t+1}\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t C_t^\sigma N_t^{1-\varepsilon-\sigma} \\
\text{s.t. } & C_t + K_{t+1} = F(K_t, N_t - q_t N_{t+1}; t) + (1 - \delta)K_t, \\
& L_t = N_t - q_t N_{t+1}, \\
& F(K_t, L_t; t) \equiv \max_{0 \leq K_{1t} \leq K_t, 0 \leq L_{1t} \leq L_t} \left\{ A_{1t} K_{1t}^\phi L_{1t}^\mu \Lambda^{1-\phi-\mu} + A_{2t} (K_t - K_{1t})^\theta (L_t - L_{1t})^{1-\theta} \right\}, \\
& \text{Nonnegativity, } K_0, N_0 \text{ given.}
\end{aligned}$$

7.1 Solving the model

All of the propositions from section 5 apply here except the equations must be replaced appropriately. In this section we derive all of the relevant equations for the case of the Barro and Becker utility.

We use the following notation for the factor prices:

$$\begin{aligned}
r_t & \equiv F_1(K_t, N_t - q_t N_{t+1}, t) \\
w_t & \equiv F_2(K_t, N_t - q_t N_{t+1}, t)
\end{aligned}$$

F.O.C.'s

$$\begin{aligned}
[C_t] & : \sigma \beta^t N_t^{1-\varepsilon-\sigma} C_t^{\sigma-1} = \lambda_t, \quad \sigma \beta^{t+1} N_{t+1}^{1-\varepsilon-\sigma} C_{t+1}^{\sigma-1} = \lambda_{t+1} \\
[K_{t+1}] & : \lambda_t = \lambda_{t+1} (r_{t+1} + 1 - \delta) \\
[N_{t+1}] & : (1 - \varepsilon - \sigma) \beta^{t+1} C_{t+1}^\sigma N_{t+1}^{-\varepsilon-\sigma} = \lambda_t q_t w_t - \lambda_{t+1} w_{t+1}
\end{aligned}$$

The first order and feasibility conditions are:

$$\begin{aligned}
\left(\frac{C_{t+1}}{C_t} \right)^{1-\sigma} & = \left(\frac{N_{t+1}}{N_t} \right)^{1-\varepsilon-\sigma} \beta (r_{t+1} + 1 - \delta) \\
\frac{(1 - \varepsilon - \sigma) \beta}{\sigma} \frac{C_{t+1}}{N_{t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{\sigma-1} \left(\frac{N_{t+1}}{N_t} \right)^{1-\varepsilon-\sigma} & = q_t w_t - \frac{w_{t+1}}{r_{t+1} + 1 - \delta} \\
C_t + K_{t+1} & = A_t K_t^\theta (N_t - q_t N_{t+1})^{1-\theta} + (1 - \delta) K_t
\end{aligned}$$

In per capita variables: $\{c_t, k_{t+1}, N_{t+1}\}_{t=0}^\infty$

$$\begin{aligned}
\left(\frac{c_{t+1}}{c_t} \right)^{1-\sigma} n_t^\varepsilon & = \beta (r_{t+1} + 1 - \delta) \\
\frac{(1 - \varepsilon - \sigma)}{\sigma} c_{t+1} & = (r_{t+1} + 1 - \delta) q_t w_t - w_{t+1} \\
c_t + k_{t+1} n_t & = A_t k_t^\theta (1 - q n_t)^{1-\theta} + (1 - \delta) k_t
\end{aligned}$$

Compare these conditions to the corresponding conditions for the Solow model with Lucas utility

$$\begin{aligned}
\frac{c_{t+1}}{c_t} n_t & = \beta (r_{t+1} + 1 - \delta) \\
\frac{(1 - \alpha - \beta)}{\alpha \beta} c_{t+1} & = (r_{t+1} + 1 - \delta) q_t w_t - w_{t+1} \\
c_t + k_{t+1} n_t & = A_t k_t^\theta (1 - q n_t)^{1-\theta} + (1 - \delta) k_t \\
\frac{c_{t+1}}{c_t} n_t & = \beta (r_{t+1} + 1 - \delta) \\
\frac{(1 - \alpha - \beta)}{\alpha} c_t / n_t & = q_t w_t - \frac{w_{t+1}}{(r_{t+1} + 1 - \delta)}
\end{aligned}$$

7.2 Deriving Malthus-Solow BGP conditions

On the BGP, we have

$$\gamma^{1-\sigma} = \frac{\beta}{n^\varepsilon} [r_{t+1} + 1 - \delta] \quad (127)$$

$$\frac{(1-\varepsilon-\sigma)}{\sigma} c_{t+1} = (r_{t+1} + 1 - \delta) q_t w_t - w_{t+1} \quad (128)$$

$$\phi A_{10} \gamma_1^t k_{1t}^{\phi-1} l_{1t}^\mu \lambda_t^{1-\phi-\mu} = \theta A_{20} \gamma_2^t (k_t - k_{1t})^{\theta-1} (1 - l_{1t} - qn)^{1-\theta} = r_t \quad (129)$$

$$\mu A_{10} \gamma_1^t k_{1t}^\phi l_{1t}^{\mu-1} \lambda_t^{1-\phi-\mu} = (1-\theta) A_{20} \gamma_2^t (k_t - k_{1t})^\theta (1 - l_{1t} - qn)^{-\theta} = w_t \quad (130)$$

$$c_t + k_{t+1}n = A_{10} \gamma_1^t k_{1t}^\phi (l_{1t})^\mu (\lambda_t)^{1-\phi-\mu} + A_{20} \gamma_2^t (k_t - k_{1t})^\theta (1 - l_{1t} - qn)^{1-\theta} + (1-\delta)k_t \quad (131)$$

Note that on the BGP, k_{1t} must grow at the same rate as k_t , hence $\rho_k = k_{1t}/k_t$ must stay constant. Also l_{1t} must stay constant, denote it by l_1 . From the first equation we have r_t must be constant, call it r .

$$\gamma^{1-\sigma} = \frac{\beta}{n^\varepsilon} [r + 1 - \delta] \quad (132)$$

$$\frac{(1-\varepsilon-\sigma)}{\sigma} c_{t+1} = (r + 1 - \delta) q_t w_t - w_{t+1} \quad (133)$$

$$\phi A_{10} \gamma_1^t (\rho_k k_t)^{\phi-1} l_1^\mu \lambda_t^{1-\phi-\mu} = \theta A_{20} \gamma_2^t (1 - \rho_k)^{\theta-1} k_t^{\theta-1} (1 - l_1 - qn)^{1-\theta} = r \quad (134)$$

$$\mu A_{10} \gamma_1^t (\rho_k k_t)^\phi l_1^{\mu-1} \lambda_t^{1-\phi-\mu} = (1-\theta) A_{20} \gamma_2^t (1 - \rho_k)^\theta k_t^\theta (1 - l_1 - qn)^{-\theta} = w \quad (135)$$

$$c_t + k_{t+1}n = A_1 \gamma_1^t k_t^\phi \rho_k^\phi l_1^\mu \lambda_t^{1-\phi-\mu} + A_{20} \gamma_2^t (1 - \rho_k)^\theta k_t^\theta (1 - l_1 - qn)^{1-\theta} + (1-\delta)k_t \quad (136)$$

So, marginal products of capital in both sectors are constant. MP_K in Malthusian technology being constant gives

$$\begin{aligned} r &= \phi A_{10} \gamma_1^t (\rho_k k_t)^{\phi-1} l_1^\mu \lambda_t^{1-\phi-\mu} \\ \gamma_1 &= n^{1-\phi-\mu} \gamma^{1-\phi} \end{aligned}$$

and MP_K in Solow technology being constant implies $\gamma_2^t k_t^{\theta-1}$ must be constant, so that

$$\gamma_2 = \gamma^{1-\theta}$$

Combining these two equations pins down the BGP growth rates of population and per capita variables:

$$\boxed{\gamma = \gamma_2^{\frac{1}{1-\theta}}, \quad n = \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}} \right)^{\frac{1}{1-\phi-\mu}}}$$

Further, c_t and w_t must grow at the same rate from the second equation. Denote $\eta = \frac{c_t}{w_t}$. From feasibility c_t and k_{t+1} must grow at the same rate. Denote $\rho = \frac{c_t}{k_t}$.

Malthusian output can be rewritten as $y_{1t} = \frac{r k_{1t}}{\phi} = \frac{r \rho_k k_t}{\phi}$ or as $y_{1t} = \frac{w_t l_1}{\mu}$. Hence, we can solve for w_t in terms of k_t

$$\begin{aligned} \frac{r \rho_k k_t}{\phi} &= \frac{w_t l_1}{\mu} \\ w_t &= \frac{\mu r \rho_k k_t}{\phi l_1} \end{aligned}$$

Then we get a relationship between η and ρ

$$\eta = \frac{c_t}{w_t} = \frac{c_t \phi l_1}{\mu r \rho_k k_t} = \frac{\rho \phi l_1}{\mu r \rho_k}$$

Hence, we can rewrite the second equation as

$$\frac{(1 - \varepsilon - \sigma)}{\sigma} \frac{\rho \phi l_1}{\mu r \rho_k} = \frac{q(r + 1 - \delta)}{\gamma} - 1$$

Next define $\rho_y = \frac{y_{1t}}{y_t}$, then we can rewrite the third equation as

$$\begin{aligned} \frac{\phi y_{1t}}{\rho_k k_t} &= \frac{\theta y_{2t}}{(1 - \rho_k) k_t} \\ \frac{\phi \rho_y y_t}{\rho_k} &= \frac{\theta (1 - \rho_y) y_t}{(1 - \rho_k)} \\ \frac{\phi \rho_y}{(1 - \rho_y)} &= \frac{\theta \rho_k}{(1 - \rho_k)} \end{aligned}$$

and the forth equation as

$$\begin{aligned} \frac{\mu y_{1t}}{l_1} &= \frac{(1 - \theta) y_{2t}}{(1 - l_1 - qn)} \\ \frac{\mu \rho_y}{(1 - \rho_y)} &= \frac{(1 - \theta) l_1}{(1 - l_1 - qn)} \end{aligned}$$

Finally, we rewrite the feasibility condnion as

$$\begin{aligned} c_t + k_{t+1}n &= y_t + (1 - \delta)k_t \\ c_t + k_{t+1}n &= \frac{y_{1t}}{\rho_y} + (1 - \delta)k_t \\ c_t + k_{t+1}n &= \frac{r \rho_k k_t}{\rho_y \phi} + (1 - \delta)k_t \\ \rho + \gamma n &= \frac{r \rho_k}{\rho_y \phi} + (1 - \delta) \end{aligned}$$

Hence, the system of equations becomes

$$\gamma = \gamma_2^{\frac{1}{1-\theta}} \tag{137}$$

$$n = \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}} \right)^{\frac{1}{1-\phi-\mu}} \tag{138}$$

$$\gamma^{1-\sigma} = \frac{\beta}{n^\varepsilon} [r + 1 - \delta] \tag{139}$$

$$\frac{(1 - \varepsilon - \sigma)}{\sigma} \frac{\rho \phi l_1}{\mu r \rho_k} = \frac{q(r + 1 - \delta)}{\gamma} - 1 \tag{140}$$

$$\frac{\theta \rho_k}{(1 - \rho_k)} = \frac{\phi \rho_y}{(1 - \rho_y)} \tag{141}$$

$$\frac{\mu \rho_y}{(1 - \rho_y)} = \frac{(1 - \theta) l_1}{(1 - l_1 - qn)} \tag{142}$$

$$\rho + \gamma n = \frac{r \rho_k}{\rho_y \phi} + (1 - \delta) \tag{143}$$

where the unknowns are $\gamma, n, r, l_1, \rho, \rho_k, \rho_y$. ($\rho = \frac{c}{k}, \rho_k = \frac{k_1}{k}, \rho_y = \frac{y_1}{y}$).
Next solve for the steady state in detrended variables.

7.3 Calibration of the model under the assumption of the Barro and Becker parental utility

Notice that we can also rewrite these equations in terms of available moments $(\frac{l_1}{l}, \frac{y_1}{y}, \frac{wl}{y}, \frac{rk}{y})$ and parameters only.

First, labor share is available and hence it directly pins down μ : $\frac{wl}{y} = \frac{\mu y_1}{l_1} \frac{l}{y} \Rightarrow \mu$

Equation six is $w_t = \frac{\mu y_{1t}}{l_1} = \frac{(1-\theta)y_{2t}}{(1-l_1-qn)}$. Combine it with an algebraic identity to get

$$\begin{aligned} 1 - qn &= l_1 + 1 - l_1 - qn \\ wl &= wl_1 + w(1 - l_1 - qn) \\ \frac{wl}{y} &= \frac{\mu y_1}{l_1} \frac{l_1}{y} + \frac{(1-\theta)y_2}{(1-l_1-qn)} \frac{(1-l_1-qn)}{y} \\ \frac{wl}{y} &= \frac{\mu y_1}{y} + \frac{(1-\theta)y_2}{y} \end{aligned}$$

This pins down θ . Next, equation five is just $r \equiv \frac{\phi y_1}{\rho_k k} = \frac{\theta y_2}{(1-\rho_k)k}$. Combining it with an algebraic identity we get

$$\begin{aligned} k &= \rho_k k + (1 - \rho_k) k \\ rk &= r \rho_k k + r(1 - \rho_k) k \\ rk &= \frac{\phi y_1}{\rho_k k} \rho_k k + \frac{\theta y_2}{(1 - \rho_k) k} (1 - \rho_k) k \\ rk &= \phi y_1 + \theta y_2 \\ \frac{rk}{y} &= \frac{\phi y_1}{y} + \frac{\theta y_2}{y} \end{aligned}$$

This allows us to get γ_1 and γ_2 and give prediction to γ and n .

We then use the moment qn and the newly obtained n to get $a/\pi + b$. Separately employing the assumption on a/b and π we calibrate a and b . Finally, we combine equations four and seven

$$\frac{(1 - \varepsilon - \sigma)}{\sigma} \frac{\rho \phi l_1}{\mu r \rho_k} = \frac{q(r + 1 - \delta)}{\gamma} - 1 \quad (144)$$

$$\rho + \gamma n = \frac{y}{k} + (1 - \delta) \quad (145)$$

to get

$$\frac{(1 - \varepsilon - \sigma)}{\sigma} \frac{\phi l_1}{\mu r \rho_k} \left(r \frac{y}{rk} + (1 - \delta) - \gamma n \right) = \frac{q(r + 1 - \delta)}{\gamma} - 1 \quad (146)$$

$$\frac{(1 - \varepsilon - \sigma)(1 - qn)}{\sigma \mu} \frac{\phi}{r \rho_k} \frac{l_1}{(1 - qn)} \left(r \frac{y}{rk} + (1 - \delta) - \gamma n \right) = \frac{qn(r + 1 - \delta)}{\gamma n} - 1 \quad (147)$$

Since

$$\begin{aligned} r &= \frac{\phi y_1}{k_1} \\ \frac{k}{y_1} &= \frac{\phi}{r \rho_k} \end{aligned}$$

and hence, the above becomes

$$\frac{(1 - \varepsilon - \sigma)(1 - qn)}{\sigma \mu} \frac{1}{r} \frac{y}{y_1} \frac{rk}{y} \frac{l_1}{(1 - qn)} \left(r \frac{y}{rk} + (1 - \delta) - \gamma n \right) = \frac{qn(r + 1 - \delta)}{\gamma n} - 1$$

which pins down $\frac{1-\varepsilon-\sigma}{\sigma}$. If we set one of them, say ε , then we know the other, σ , from this equation. We can think of solving the last equation as solving 2 equations in two unknowns: ρ and σ .

Then $\gamma^{1-\sigma} = \frac{\beta}{n^\varepsilon} [r + 1 - \delta]$ can be used together with $r + 1 - \delta$ in the data to get β . Hence, we solve the following equations

$$\frac{wl}{y} = \frac{\mu y_1}{l_1} \frac{l}{y} \quad (148)$$

$$\frac{wl}{y} = \frac{\mu y_1}{y} + \frac{(1-\theta)y_2}{y} \quad (149)$$

$$\frac{rk}{y} = \frac{\phi y_1}{y} + \frac{\theta y_2}{y} \quad (150)$$

$$\gamma = \gamma_2^{\frac{1}{1-\theta}} \quad (151)$$

$$n = \left(\gamma_1 \gamma_2^{-\frac{1-\phi}{1-\theta}} \right)^{\frac{1}{1-\phi-\mu}} \quad (152)$$

$$\frac{qn}{n} = \frac{b}{\pi} \frac{a}{b} + b \quad (153)$$

$$a = b \frac{a}{b} \quad (154)$$

$$\frac{(1-\varepsilon-\sigma)(1-qn)}{\sigma\mu} \frac{1}{r} \frac{y}{y_1} \frac{rk}{y} \frac{l_1}{(1-qn)} \rho = \frac{qn(r+1-\delta)}{\gamma n} - 1 \quad (155)$$

$$\rho + \gamma n = r \frac{y}{rk} + (1-\delta) \quad (156)$$

$$\gamma^{1-\sigma} = \frac{\beta}{n^\varepsilon} [r + 1 - \delta] \quad (157)$$

So, π , δ , γ_1 , γ_2 (although the last two can be pinned down only after ϕ, μ, θ are determined) are directly pinned down in the data. Also, ε is set.

10 equations, 10 unknowns: 7 parameters $\mu, \phi, \theta, \beta, a, b, \sigma$, and 3 moments that we do not take from the data: ρ, γ, n . Moments used $\frac{wl}{y}, \frac{rk}{y}, \frac{y_1}{y}, \frac{l_1}{l}, qn, \frac{a}{b}, r$. We have 7 moments and 7 parameters.

8 Effects of TFP and Labor Supply Changes on Structural Change

8.1 Changes in Productivity

In this subsection, we analyze the effect of changes in productivity on inputs allocation in a two sector economy. In particular, we focus on the role that preferences play in determining the direction of resource reallocation.

We describe a simple static general equilibrium model economy, with two sectors. The numeraire is good 2, and the prices of labor and good 1 are quoted in units of good 2. The Households solve

$$\max_{c_1, c_2} u(c_1, c_2)$$

$$s.t.$$

$$pc_1 + c_2 = w + \pi_1 + \pi_2$$

Firms solve

$$\max_{L_1} \pi_1 = pA_1f(L_1) - wL_1$$

$$\max_{L_2} \pi_2 = A_2g(L_2) - wL_2$$

Market clearing

$$[\text{Labor mkt.}] : L_1 + L_2 = 1$$

$$[\text{Good 1}] : c_1 = A_1f(L_1)$$

$$[\text{Good 2}] : c_2 = A_2g(L_2)$$

Equilibrium: prices (p^*, w^*) and allocation $(c_1^*, c_2^*, L_1^*, L_2^*, \pi_1^*, \pi_2^*)$, such that (i) given the prices (c_1^*, c_2^*) solves the household's problem, (ii) given the prices $(L_1^*, L_2^*, \pi_1^*, \pi_2^*)$ solves the firms' problems, and (iii) markets are cleared.

The above equilibrium can be computed by solving the social planner's problem

$$\begin{aligned} \max_{c_1, c_2} & u(c_1, c_2) \\ \text{s.t.} & \\ c_1 &= A_1 f(L_1) \\ c_2 &= A_2 g(L_2) \\ L_1 + L_2 &= 1 \end{aligned}$$

If the solution is interior $(c_1^*, c_2^* > 0)$, then the relative price of c_1 can be inferred from

$$\frac{u_1(c_1^*, c_2^*)}{u_2(c_1^*, c_2^*)} = p$$

Wage can be inferred from $pA_1f'(L_1^*) = w$, if $L_1^* > 0$ or from $A_2g'(L_2^*) = w$, if $L_2^* > 0$.

Corner solution means that only one sector is producing. Corner solution is possible if and only if two conditions hold: (i) the utility does not satisfy Inada conditions, and (ii) at least one of the production functions does not satisfy Inada conditions. If either the utility, or both production functions, satisfy the Inada conditions, then we must have interior solution $(c_1^*, c_2^* > 0)$. To see this, consider the Marginal Rate of Substitution and Marginal Rate of Transformation.

$$\begin{aligned} MRS &\equiv \frac{dc_2}{dc_1} (\text{on IC}) = \frac{u_1(c_1^*, c_2^*)}{u_2(c_1^*, c_2^*)} \\ MRT &\equiv \frac{dc_2}{dc_1} (\text{on PPF}) = \frac{A_2g'(L_2)}{A_1f'(L_1)} \end{aligned}$$

If u satisfies the Inada conditions, then MRS attains all the values between 0 and ∞ , and even if the PPF is linear, tangency occurs at an interior point of the PPF. If both production functions satisfy Inada conditions, then the MRT attains all the values between 0 and ∞ , and even if indifference curves are linear, a tangency will occur at the interior of the PPF.

Example of corner solution. Suppose that $u(c_1, c_2) = c_1 + \beta c_2$, thus the utility does not satisfy Inada conditions. The production functions are

$$\begin{aligned} c_1 &= A_1 L_1^\theta, \quad 0 < \theta < 1 \\ c_2 &= A_2 L_2 \end{aligned}$$

i.e., the first one satisfies Inada, and the 2nd doesn't. In this case, $MRS = 1/\beta$ and $MRT = A_2/(\theta A_1 L_1^{\theta-1}) = L_1^{1-\theta} \left(\frac{A_2}{\theta A_1} \right)$. The slope of the PPF at bottom end (when $c_2 = 0$) is $MRT = \frac{A_2}{\theta A_1}$. We have a corner solution with only c_1 being produced if and only if $1/\beta \geq \frac{A_2}{\theta A_1}$. This is analogous to the Malthus Only case in our paper. The solution is interior if $1/\beta < \frac{A_2}{\theta A_1}$.

We study sectoral shifts as a result of technological change, in the case of interior solution.

$$\begin{aligned} MRS &= MRT \\ \frac{u_1(c_1, c_2)}{u_2(c_1, c_2)} &= \frac{A_2g'(L_2)}{A_1f'(L_1)} \end{aligned}$$

Substituting the production functions and the labor constraint, we have one equation with one unknown L_1 . If the solution is unique, then this equation uniquely determines the labor allocation in this economy.

$$\begin{aligned} \frac{u_1(A_1 f(L_1), A_2 g(1 - L_1))}{u_2(A_1 f(L_1), A_2 g(1 - L_1))} &= \frac{A_2 g'(1 - L_1)}{A_1 f'(L_1)} \\ A_1 f'(L_1) u_1(A_1 f(L_1), A_2 g(1 - L_1)) - A_2 g'(1 - L_1) u_2(A_1 f(L_1), A_2 g(1 - L_1)) &= 0 \end{aligned}$$

We are interested in the signs of $\partial L_1/\partial A_1$ and $\partial L_1/\partial A_2$.

$$\frac{\partial L_1}{\partial A_1} = - \frac{f' [u_1 + A_1 u_{11} f] - A_2 g' u_{21} f}{A_1 [f'' u_1 + f' \cdot (u_{11} A_1 f' - u_{12} A_2 g')] - A_2 [-g'' u_2 + g' \cdot (u_{21} A_1 f' - u_{22} A_2 g')]}$$

It is hard to see anything from this condition. If utility is additively separable, then $u_{12} = u_{21} = 0$.

$$\frac{\partial L_1}{\partial A_1} = - \frac{f' [u_1 + A_1 u_{11} f]}{A_1 [f'' u_1 + f' \cdot u_{11} A_1 f'] + A_2 [g'' u_2 + g' u_{22} A_2 g']}$$

Even with the separability assumption, we can't say much about the sign of $\partial L_1/\partial A_1$. For instance, the sign of the numerator depends on $\text{sign}(u_1 + A_1 u_{11} f)$. If $u_1 > 0$ and $u_{11} < 0$, there is no way to determine this sign. Moreover, an important CES utility class is not separable.

Suppose that the two goods are perfect substitutes, so the utility is linear, i.e. $u_{11} = u_{22} = 0$, and we have

$$\frac{\partial L_1}{\partial A_1} = - \frac{f' u_1}{A_1 f'' u_1 + A_2 g'' u_2} > 0$$

The sign is always positive if production functions are strictly concave. This is the only general result that we can obtain. Thus, with perfect substitutes, an increase in productivity in sector i will always pull the labor to that industry.

Important. Apriori, we have no idea how to model the preferences between the two goods. However, when we measure the TFP's in the two sectors, we find that the TFP in the Solow sector increased more. If we assume that the two goods are perfect substitutes, then we guarantee that the labor will reallocate towards the Solow sector. Notice that our estimates of TFP are *independent* of assumptions on preferences. Thus, our estimates of the TFP's impose restrictions on the kind of preferences that we need in order to generate the sectoral transition.

8.1.1 CES utility, Cobb-Douglas production

Suppose that

$$\begin{aligned} u(c_1, c_2) &= [\alpha c_1^\rho + (1 - \alpha) c_2^\rho]^{1/\rho}, \quad \rho \leq 1 \\ c_1 &= A_1 L_1^{\theta_1}, \quad 0 < \theta_1 \leq 1 \\ c_2 &= A_2 L_2^{\theta_2}, \quad 0 < \theta_2 \leq 1 \\ L_1 + L_2 &= 1 \end{aligned}$$

The elasticity of substitution between c_1 and c_2 is $1/(1 - \rho)$. Recall that

$\rho = 1$	Perfect substitutes
$0 < \rho < 1$	Substitutes
$\rho = 1$	Cobb-Douglas
$-\infty < \rho < 0$	Complements
$\rho = -\infty$	Perfect complements

Assuming interior solution, we have

$$\begin{aligned} MRS &= MRT \\ \left(\frac{\alpha}{1 - \alpha} \right) \frac{c_1^{\rho-1}}{c_2^{\rho-1}} &= \frac{\theta_2 A_2 L_2^{\theta_2-1}}{\theta_1 A_1 L_1^{\theta_1-1}} \\ \left(\frac{\alpha}{1 - \alpha} \right) \left(\frac{A_2 (1 - L_1)^{\theta_2}}{A_1 L_1^{\theta_1}} \right)^{1-\rho} &= \frac{\theta_2 A_2 (1 - L_1)^{\theta_2-1}}{\theta_1 A_1 L_1^{\theta_1-1}} \\ \left(\frac{\alpha}{1 - \alpha} \right) \frac{(1 - L_1)^{\theta_2(1-\rho)}}{L_1^{\theta_1(1-\rho)}} &= \left(\frac{A_2}{A_1} \right)^\rho \frac{\theta_2 (1 - L_1)^{\theta_2-1}}{\theta_1 L_1^{\theta_1-1}} \\ \Psi(L_1, A_1, A_2) &\equiv \frac{(1 - L_1)^{1-\rho\theta_2}}{L_1^{1-\rho\theta_1}} - \frac{\theta_2}{\theta_1} \left(\frac{1 - \alpha}{\alpha} \right) \left(\frac{A_2}{A_1} \right)^\rho = 0 \end{aligned}$$

We are interested in $\text{sign}(\partial L_1 / \partial A_1)$, i.e.

$$\frac{\partial L_1}{\partial A_1} = -\frac{\Psi_2}{\Psi_1}$$

The sign of Ψ_2 depends on ρ in the following manner

$$\text{sign}(\Psi_2) \begin{cases} > 0 & \text{if } \rho > 0 \\ = 0 & \text{if } \rho = 0 \\ < 0 & \text{if } \rho < 0 \end{cases}$$

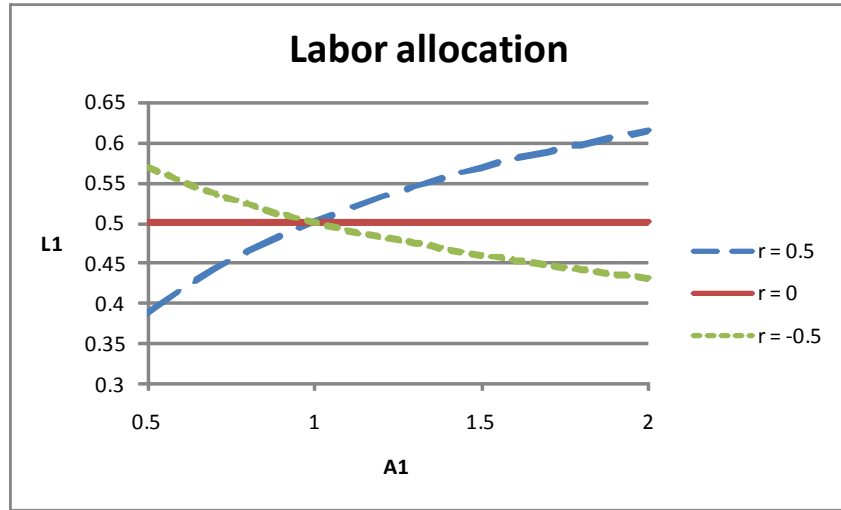
Now we turn to the sign of $\Psi_1(L_1, A_1, A_2)$.

$$\begin{aligned} \Psi_1 &= -(1 - \rho\theta_2)(1 - L_1)^{-\rho\theta_2} L_1^{\rho\theta_1-1} + (1 - L_1)^{1-\rho\theta_2} (\rho\theta_1 - 1) L_1^{\rho\theta_1-2} < 0 \\ &(\rho\theta_2 - 1)L_1 + (1 - L_1)(\rho\theta_1 - 1) < 0 \end{aligned}$$

Notice that $\rho\theta_i - 1$ is always negative when $\rho < 1$ and $0 < \theta_i \leq 1$. Thus, the $\text{sign}(\partial L_1 / \partial A_1)$ is determined by $\text{sign}(\Psi_2)$, i.e.

$$\text{sign}(\partial L_1 / \partial A_1) \begin{cases} > 0 & \text{if } \rho > 0 & \text{Substitutes} \\ = 0 & \text{if } \rho = 0 & \text{Cobb-Douglas} \\ < 0 & \text{if } \rho < 0 & \text{Complements} \end{cases}$$

When the two goods are complements (it is hard to substitute manufacturing goods for food), we must have a *push effect* of productivity in a sector on the labor allocated to that sector. If the goods are substitutes, there is a *pull effect* of productivity in a sector on the labor allocated to that sector. If the utility is Cobb-Douglas, labor allocation between the two sectors is independent of the productivities in the two sectors. The next figure shows the labor allocation for $\theta_1 = \theta_2 = 0.5$ and 3 cases of elasticity of substitution: $\rho = 0.5$, $\rho = 0$, $\rho = -0.5$.



8.1.2 Stone-Geary utility, Cobb-Douglas production

Suppose that

$$\begin{aligned} u(c_1, c_2) &= \alpha \ln(c_1 - \tilde{c}_1) + (1 - \alpha) \ln(c_2 - \tilde{c}_2), \quad \tilde{c}_1, \tilde{c}_2 \geq 0 \\ c_1 &= A_1 L_1^{\theta_1}, \quad 0 < \theta_1 \leq 1 \\ c_2 &= A_2 L_2^{\theta_2}, \quad 0 < \theta_2 \leq 1 \\ L_1 + L_2 &= 1 \end{aligned}$$

These preferences reduce to Cobb-Douglas when $\tilde{c}_1 = \tilde{c}_2 = 0$. Assuming interior solution, we have

$$\begin{aligned}
MRS &= MRT \\
\left(\frac{\alpha}{1-\alpha}\right) \frac{c_2 - \tilde{c}_2}{c_1 - \tilde{c}_1} &= \frac{\theta_2 A_2 L_2^{\theta_2-1}}{\theta_1 A_1 L_1^{\theta_1-1}} \\
\left(\frac{\alpha}{1-\alpha}\right) \frac{A_2 (1-L_1)^{\theta_2} - \tilde{c}_2}{A_1 L_1^{\theta_1} - \tilde{c}_1} &= \left(\frac{\theta_2}{\theta_1}\right) \frac{A_2 (1-L_1)^{\theta_2-1}}{A_1 L_1^{\theta_1-1}} \\
\left(\frac{\alpha}{1-\alpha}\right) \frac{A_2 (1-L_1)^{\theta_2} - \tilde{c}_2}{A_2 (1-L_1)^{\theta_2-1}} &= \left(\frac{\theta_2}{\theta_1}\right) \frac{A_1 L_1^{\theta_1} - \tilde{c}_1}{A_1 L_1^{\theta_1-1}} \\
\left(\frac{\alpha}{1-\alpha}\right) \left[\frac{A_2 (1-L_1)^{\theta_2}}{A_2 (1-L_1)^{\theta_2-1}} - \frac{\tilde{c}_2}{A_2 (1-L_1)^{\theta_2-1}} \right] &= \left(\frac{\theta_2}{\theta_1}\right) \left[\frac{A_1 L_1^{\theta_1}}{A_1 L_1^{\theta_1-1}} - \frac{\tilde{c}_1}{A_1 L_1^{\theta_1-1}} \right] \\
\left(\frac{\alpha}{1-\alpha}\right) \left[1 - L_1 - \frac{\tilde{c}_2 (1-L_1)^{1-\theta_2}}{A_2} \right] &= \left(\frac{\theta_2}{\theta_1}\right) \left[L_1 - \frac{\tilde{c}_1 L_1^{1-\theta_1}}{A_1} \right] \\
\left(\frac{\alpha}{1-\alpha}\right) (1-L_1) - \left(\frac{\alpha}{1-\alpha}\right) \frac{\tilde{c}_2}{A_2} (1-L_1)^{1-\theta_2} &= \left(\frac{\theta_2}{\theta_1}\right) L_1 - \left(\frac{\theta_2}{\theta_1}\right) \frac{\tilde{c}_1}{A_1} L_1^{1-\theta_1} \\
\left(\frac{\alpha}{1-\alpha}\right) + \left(\frac{\theta_2}{\theta_1}\right) \frac{\tilde{c}_1}{A_1} L_1^{1-\theta_1} - \left(\frac{\alpha}{1-\alpha}\right) \frac{\tilde{c}_2}{A_2} (1-L_1)^{1-\theta_2} - L_1 \left[\frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} \right] &= 0
\end{aligned} \tag{158}$$

Let

$$\begin{aligned}
\Psi(L_1, A_1, A_2) &= \left(\frac{\alpha}{1-\alpha}\right) + \left(\frac{\theta_2}{\theta_1}\right) \frac{\tilde{c}_1}{A_1} L_1^{1-\theta_1} - \left(\frac{\alpha}{1-\alpha}\right) \frac{\tilde{c}_2}{A_2} (1-L_1)^{1-\theta_2} - \phi L_1 \\
\text{where } \phi &= \frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} > 0
\end{aligned} \tag{159}$$

First we show that Ψ_1 is always negative.

$$\begin{aligned}
\Psi_1 &= (1-\theta_1) \left(\frac{\theta_2}{\theta_1}\right) \frac{\tilde{c}_1}{A_1} L_1^{1-\theta_1} + (1-\theta_2) \left(\frac{\alpha}{1-\alpha}\right) \frac{\tilde{c}_2}{A_2} (1-L_1)^{-\theta_2} - \phi \\
\Psi_1 &= (1-\theta_1) \left(\frac{\theta_2}{\theta_1}\right) \frac{\tilde{c}_1}{c_1} + (1-\theta_2) \left(\frac{\alpha}{1-\alpha}\right) \frac{\tilde{c}_2}{c_2} - \phi < 0
\end{aligned}$$

Consider and upper bound on Ψ_1 when $c_1 \searrow \tilde{c}_1$ and $c_2 \searrow \tilde{c}_2$. We show that this upper bound is always negative.

$$\begin{aligned}
\bar{\Psi}_1 &= (1-\theta_1) \left(\frac{\theta_2}{\theta_1}\right) + (1-\theta_2) \left(\frac{\alpha}{1-\alpha}\right) - \frac{\alpha}{1-\alpha} - \frac{\theta_2}{\theta_1} \\
&= \frac{\theta_2 (1-\theta_1) - \theta_2}{\theta_1} + \frac{\alpha (1-\theta_2) - \alpha}{1-\alpha} \\
&= \frac{\theta_2 (1-\theta_1) - \theta_2}{\theta_1} + \frac{\alpha (1-\theta_2) - \alpha}{1-\alpha} \\
&= \frac{\theta_2 - \theta_1 \theta_2 - \theta_2}{\theta_1} + \frac{\alpha - \alpha \theta_2 - \alpha}{1-\alpha} \\
&= -\theta_2 - \frac{\alpha \theta_2}{1-\alpha} < 0
\end{aligned}$$

Next, notice that

$$\begin{aligned}
\Psi_2 &< 0 \quad \text{if } \tilde{c}_1 > 0 \\
\Psi_2 &= 0 \quad \text{if } \tilde{c}_1 = 0
\end{aligned}$$

Thus

$$\frac{\partial L_1}{\partial A_1} = -\frac{\Psi_2}{\Psi_1} \begin{cases} < 0 & \text{if } \tilde{c}_1 > 0 \\ = 0 & \text{if } \tilde{c}_1 = 0 \end{cases}$$

To summarize the results, the presence of minimum consumption requirement in industry i creates a push effect of productivity growth on labor employed in that sector. This means that if only one good has minimum consumption requirement, then an increase in that sector's productivity will push the labor away from that sector. An increase in productivity of a sector which does not have minimum consumption requirement, does not have any effect on labor allocation between the two sectors. To convince yourself, observe equation (159) which determines the allocation of labor between the two sectors. If $\tilde{c}_1 = 0$, then A_1 does not have any effect of labor allocation between the two sectors. If $\tilde{c}_2 = 0$, then A_2 does not have any effect of labor allocation between the two sectors. Interestingly, if both goods have minimum consumption requirement and both productivities go up, then labor allocation can change in either direction

8.1.3 Quasi-linear utility, Cobb-Douglas production

Suppose that

$$\begin{aligned} u(c_1, c_2) &= \gamma \ln(c_1) + c_2 \\ c_1 &= A_1 L_1^{\theta_1}, \quad 0 < \theta_1 \leq 1 \\ c_2 &= A_2 L_2^{\theta_2}, \quad 0 < \theta_2 \leq 1 \\ L_1 + L_2 &= 1 \end{aligned}$$

Assuming interior solution, we have

$$\begin{aligned} MRS &= MRT \\ \frac{\gamma}{c_1} &= \frac{\theta_2 A_2 L_2^{\theta_2-1}}{\theta_1 A_1 L_1^{\theta_1-1}} \\ \frac{\gamma}{A_1 L_1^{\theta_1}} &= \frac{\theta_2 A_2 L_2^{\theta_2-1}}{\theta_1 A_1 L_1^{\theta_1-1}} \\ \frac{\gamma}{L_1^{\theta_1}} &= \frac{\theta_2 A_2 L_2^{\theta_2-1}}{\theta_1 L_1^{\theta_1-1}} \end{aligned}$$

Clearly, labor allocation is independent of A_1 . Lets examine the dependence on A_2 . Simplifying the last term, gives

$$\gamma \frac{(1 - L_1)^{1-\theta_2}}{L_1} = \frac{\theta_2 A_2}{\theta_1}$$

It is clear even without taking derivatives, that an increase in A_2 leads to lower L_1 , i.e. higher L_2 . Just in case, let

$$\begin{aligned} \Psi(L_1, A_1, A_2) &= (1 - L_1)^{1-\theta_2} L_1^{-1} - \frac{\theta_2 A_2}{\theta_1 \gamma} \\ \frac{\partial L_1}{\partial A_2} &= -\frac{-(\theta_2/\theta_1)}{\left[-(1 - \theta_2)(1 - L_1)^{1-\theta_2} L_1^{-1} - \gamma(1 - L_1)^{1-\theta_2} L_1^{-2} \right]} < 0 \end{aligned}$$

To summarize, A_1 does not have any effect on labor allocation, while A_2 has a *pull effect*, i.e. higher productivity in sector 2 attracts labor into that sector.

8.1.4 Intuition

In this section we are trying to find out what features of preferences determines whether we have a push effect or pull effect. One might suspect that the elasticity of substitution is the one property responsible for the direction of resource allocation due to productivity changes. This intuition turns out to be correct in most

cases, but not always. In the CES utility, indeed, when the two goods are substitutes there is a pull effect and when they are complements there is a push effect. However, with non-constant elasticity, this does not have to happen. For example, in the Stone-Geary or quasi-linear utility cases, sometimes productivity does not have any effect on labor allocation. However, when productivity does have effect on labor allocation, the direction of the effect is consistent with economic intuition and the results obtained in the CES case. Consider the elasticity of substitution between c_1 and c_2 in the quasi-linear utility case.

$$\begin{aligned}
\eta &\equiv \frac{\% \Delta (c_2/c_1)}{\% \Delta (MRS)} = \frac{d(c_2/c_1)}{d(\gamma/c_1)} \frac{\gamma/c_1}{c_2/c_1} \\
&= \frac{dc_2 c_1 - c_2 dc_1}{d\gamma c_1 - \gamma dc_1} \frac{\gamma}{c_2} \\
&= - \frac{dc_2 c_1 - c_2 dc_1}{c_2 dc_1} \\
&= \frac{c_2 dc_1 - dc_2 c_1}{c_2 dc_1} \\
&= 1 - \underbrace{\left(\frac{dc_2}{dc_1} \right) \frac{c_1}{c_2}}_{MRS} \\
&= 1 + \frac{\gamma}{c_1} \frac{c_1}{c_2} \\
&= 1 + \frac{\gamma}{c_2} > 1
\end{aligned}$$

Thus, in the quasi-linear utility case, the two goods are *always substitutes*. However, the effects of changes in productivity are totally different from the CES case. Recall that in the CES case, when the two goods are substitutes, both $A_1 \uparrow$ and $A_2 \uparrow$ have pull effect on labor allocated to their corresponding sectors. In the quasi-linear case however $A_1 \uparrow$ has no effect and $A_2 \uparrow$ has a pull effect.

Consider the elasticity of substitution between c_1 and c_2 in the Stone-Geary utility case.

$$\begin{aligned}
\sigma &\equiv \frac{\% \Delta (c_2/c_1)}{\% \Delta (MRS)} = \frac{d(c_2/c_1)}{d\left(\left(\frac{\alpha}{1-\alpha}\right) \frac{c_2 - \tilde{c}_2}{c_1 - \tilde{c}_1}\right)} \frac{\left(\frac{\alpha}{1-\alpha}\right) \frac{c_2 - \tilde{c}_2}{c_1 - \tilde{c}_1}}{c_2/c_1} \\
&= \frac{d(c_2/c_1)}{d\left(\frac{c_2 - \tilde{c}_2}{c_1 - \tilde{c}_1}\right)} \frac{c_1}{c_2} \\
&= \frac{\left[\frac{dc_2 c_1 - c_2 dc_1}{c_1^2} \right]}{\left[\frac{dc_2 (c_1 - \tilde{c}_1) - (c_2 - \tilde{c}_2) dc_1}{(c_1 - \tilde{c}_1)^2} \right]} \left(\frac{c_2 - \tilde{c}_2}{c_1 - \tilde{c}_1} \right) \frac{c_1}{c_2} \\
&= \frac{\left[\frac{dc_2}{c_2} - \frac{dc_1}{c_1} \right]}{\left[\frac{dc_2}{(c_2 - \tilde{c}_2)} - \frac{dc_1}{(c_1 - \tilde{c}_1)} \right]} \\
&= \frac{\frac{dc_1}{c_1} \left[\frac{dc_2}{dc_1} \frac{c_1}{c_2} - 1 \right]}{\frac{dc_1}{(c_1 - \tilde{c}_1)} \left[\frac{dc_2}{dc_1} \frac{(c_1 - \tilde{c}_1)}{(c_2 - \tilde{c}_2)} - 1 \right]}
\end{aligned}$$

Use the fact that $MRS = -dc_2/dc_1$, that is, MRS is the negative of the slope of indifference curves, gives

$$\begin{aligned}
\sigma &= \frac{(c_1 - \tilde{c}_1) \left[-\left(\frac{\alpha}{1-\alpha}\right) \left(\frac{c_2 - \tilde{c}_2}{c_1 - \tilde{c}_1}\right) \frac{c_1}{c_2} - 1 \right]}{c_1 \left[-\left(\frac{\alpha}{1-\alpha}\right) - 1 \right]} \\
&= \frac{(c_1 - \tilde{c}_1) \left[1 + \left(\frac{\alpha}{1-\alpha}\right) \left(\frac{c_2 - \tilde{c}_2}{c_1 - \tilde{c}_1}\right) \frac{c_1}{c_2} \right]}{c_1 \left[1 + \left(\frac{\alpha}{1-\alpha}\right) \right]} \\
&= \frac{(c_1 - \tilde{c}_1) (1 - \alpha) \left[1 + \left(\frac{\alpha}{1-\alpha}\right) \left(\frac{c_2 - \tilde{c}_2}{c_1 - \tilde{c}_1}\right) \frac{c_1}{c_2} \right]}{c_1} \\
&= (1 - \alpha) \left(\frac{c_1 - \tilde{c}_1}{c_1} \right) + \alpha \left(\frac{c_2 - \tilde{c}_2}{c_2} \right) \leq 1
\end{aligned}$$

Thus, in the Stone-Geary utility case, the two goods are *always complements*. As our analysis above shows, the presence of a positive minimum consumption requirement in a sector implies a push effect of productivity increase on labor in that sector. However, if a sector does not have minimum consumption requirement, then changing productivity in that sector will not have any effect on labor allocation in the economy.

The next table summarizes the main findings. The arrows \uparrow and \downarrow indicate an increase or decrease in a variable, while a bar on top of a variable indicates that the variable remains unchanged.

Utility	Elasticity of subst.	Parameters	$A_1 \uparrow$	$A_2 \uparrow$
CES	substitutes	$0 < \rho \leq 1, \sigma > 1$	$L_1 \uparrow, L_2 \downarrow$	$L_1 \downarrow, L_2 \uparrow$
	Cobb-Douglas	$\rho = 0, \sigma = 1$	\bar{L}_1, \bar{L}_2	\bar{L}_1, \bar{L}_2
	complements	$\rho < 0, \sigma < 1$	$L_1 \downarrow, L_2 \uparrow$	$L_1 \uparrow, L_2 \downarrow$
Stone-Geary	always complements	$\tilde{c}_1 > 0, \tilde{c}_2 = 0$	$L_1 \downarrow, L_2 \uparrow$	L_1, L_2
		$\tilde{c}_1 = 0, \tilde{c}_2 > 0$	\bar{L}_1, \bar{L}_2	$L_1 \uparrow, L_2 \downarrow$
		$\tilde{c}_1 > 0, \tilde{c}_2 > 0$	$L_1 \downarrow, L_2 \uparrow$	$L_1 \uparrow, L_2 \downarrow$
Quasi-linear	always substitutes	linear in c_2	\bar{L}_1, \bar{L}_2	$L_1 \downarrow, L_2 \uparrow$
		linear in c_1	$L_1 \uparrow, L_2 \downarrow$	\bar{L}_1, \bar{L}_2

From the examples we analyzed above, we see that sometimes, changes in productivity of a sector does not have any effect on labor allocation between the two sectors. However, when change in productivity does have an effect, this effect is consistent with the basic economic intuition: (i) if goods are *complements*, higher productivity in a sector will have a *push* effect on labor in that sector, to allow higher production of the other good, and (ii) if the goods are *substitutes*, higher productivity has a *pull* effect, to lower production in the other sector.

8.2 Changes in Total Labor Endowment

In this section we analyze the effect of a change in total labor endowment on inputs allocation across sectors. In particular, we focus on the role that preferences play in determining the direction of resource reallocation. As before, we assume interior solution to begin with, i.e. both sectors operate. The labor allocation across sectors is determined by

$$\begin{aligned}
MRS &= MRT \\
\frac{u_1(c_1, c_2)}{u_2(c_1, c_2)} &= \frac{A_2 g'(L_2)}{A_1 f'(L_1)}
\end{aligned}$$

and

$$\begin{aligned}
c_1 &= A_1 f(L_1) \\
c_2 &= A_2 g(L_2) \\
L_1 + L_2 &= L
\end{aligned}$$

where L is total labor.

8.2.1 CES utility, Cobb-Douglas production

Suppose that

$$\begin{aligned} u(c_1, c_2) &= [\alpha c_1^\rho + (1 - \alpha) c_2^\rho]^{1/\rho}, \quad \rho \leq 1 \\ c_1 &= A_1 L_1^{\theta_1}, \quad 0 < \theta_1 \leq 1 \\ c_2 &= A_2 L_2^{\theta_2}, \quad 0 < \theta_2 \leq 1 \\ L_1 + L_2 &= L \end{aligned}$$

In this setup, we are interested in the signs of: (i) $\partial L_1 / \partial L$, and (ii) $\partial \lambda / \partial L$, where $\lambda = L_1 / L$ is the fraction of labor in sector 1. Assuming interior solution, we have

$$\begin{aligned} MRS &= MRT \\ \left(\frac{\alpha}{1 - \alpha} \right) \frac{c_1^{\rho-1}}{c_2^{\rho-1}} &= \frac{\theta_2 A_2 L_2^{\theta_2-1}}{\theta_1 A_1 L_1^{\theta_1-1}} \\ \left(\frac{\alpha}{1 - \alpha} \right) \left(\frac{A_2 (L - L_1)^{\theta_2}}{A_1 L_1^{\theta_1}} \right)^{1-\rho} &= \frac{\theta_2 A_2 (L - L_1)^{\theta_2-1}}{\theta_1 A_1 L_1^{\theta_1-1}} \\ \left(\frac{\alpha}{1 - \alpha} \right) \frac{(L - L_1)^{\theta_2(1-\rho)}}{L_1^{\theta_1(1-\rho)}} &= \left(\frac{A_2}{A_1} \right)^\rho \frac{\theta_2 (L - L_1)^{\theta_2-1}}{\theta_1 L_1^{\theta_1-1}} \\ \left(\frac{\alpha}{1 - \alpha} \right) \frac{(L - L_1)^{1-\rho\theta_2}}{L_1^{1-\rho\theta_1}} &= \frac{\theta_2}{\theta_1} \left(\frac{A_2}{A_1} \right)^\rho \\ \Psi(L_1, L) &= \frac{(L - L_1)^{1-\rho\theta_2}}{L_1^{1-\rho\theta_1}} - \frac{\theta_2}{\theta_1} \left(\frac{1 - \alpha}{\alpha} \right) \left(\frac{A_2}{A_1} \right)^\rho \end{aligned}$$

First, we need

$$\frac{\partial L_1}{\partial L} = -\frac{\Psi_2}{\Psi_1}$$

The sign of Ψ_2 is always positive when $\rho < 1$ and $0 < \theta_2 \leq 1$.

$$\Psi_2 = \frac{(1 - \rho\theta_2)(L - L_1)^{-\rho\theta_2}}{L_1^{1-\rho\theta_1}} > 0$$

Now we turn to the sign of Ψ_1

$$\begin{aligned} \Psi_1 &= -(1 - \rho\theta_2)(1 - L_1)^{-\rho\theta_2} L_1^{\rho\theta_1-1} + (1 - L_1)^{1-\rho\theta_2} (\rho\theta_1 - 1) L_1^{\rho\theta_1-2} < 0 \\ &(\rho\theta_2 - 1) L_1 + (1 - L_1)(\rho\theta_1 - 1) < 0 \end{aligned}$$

Which is always negative if $\rho < 1$ and $0 < \theta_i \leq 1$. Thus, $\partial L_1 / \partial L > 0$ and by symmetry $\partial L_2 / \partial L > 0$. In other words, an increase in total labor, increases the labor input in both sectors.

Now we ask what happens to the *fraction* of labor employed in each sector when total labor input goes up. Let $\lambda = L_1 / L$.

$$\begin{aligned} \Psi(\lambda, L) &= \frac{(L - \lambda L)^{1-\rho\theta_2}}{(\lambda L)^{1-\rho\theta_1}} - \frac{\theta_2}{\theta_1} \left(\frac{1 - \alpha}{\alpha} \right) \left(\frac{A_2}{A_1} \right)^\rho \\ &= L^{\rho(\theta_1 - \theta_2)} \frac{(1 - \lambda)^{1-\rho\theta_2}}{\lambda^{1-\rho\theta_1}} - \frac{\theta_2}{\theta_1} \left(\frac{1 - \alpha}{\alpha} \right) \left(\frac{A_2}{A_1} \right)^\rho \end{aligned}$$

We are interested in

$$\frac{\partial \lambda}{\partial L} = -\frac{\Psi_2}{\Psi_1}$$

Notice that Ψ_1 is always negative if $\rho < 1$ and $0 < \theta_i \leq 1$, so the sign of $\partial\lambda/\partial L$ is the same as the sign of Ψ_2 . The next table summarized the comparative statics.

$$\text{sign}(\partial\lambda/\partial L) \begin{cases} > 0 & \text{if } [\rho > 0, \theta_1 > \theta_2] \text{ or } [\rho < 0, \theta_1 < \theta_2] \\ = 0 & \text{if } \rho = 0 \text{ or } \theta_1 = \theta_2 \\ < 0 & \text{if } [\rho > 0, \theta_1 < \theta_2] \text{ or } [\rho < 0, \theta_1 > \theta_2] \end{cases}$$

We see that there are two cases where increasing labor endowment does not affect the *fraction* of labor employed in each sector. The first case is $\rho = 0$, i.e. Cobb-Douglas preferences, and the second case is $\theta_1 = \theta_2$, i.e. when the labor share in both sectors is the same. Consider now the case of $\theta_1 < \theta_2$. Both sector exhibit diminishing marginal returns to labor, but the sector with the higher θ less so. If the goods are substitutes, $\rho > 0$, and the labor endowment goes up, it is efficient to allocate more of the extra labor to the sector that "suffers less" from diminishing marginal returns. On the other hand, when the goods are complements, $\rho < 0$, the consumer wants to increase or decrease the consumption of both goods together. In this case, a greater fraction of the extra labor in the economy would be allocated towards the sector with smaller θ . One can think of the sector with smaller θ as the "weaker" sector because the returns to labor diminish faster. If the goods are complements, and the consumer wants to increase their consumption together, then the "weaker" sector "needs" more of the additional labor. When the goods are substitutes, greater portion of the extra labor goes to the "strong" sector.

8.2.2 Stone-Geary utility, Cobb-Douglas production

Suppose that

$$\begin{aligned} u(c_1, c_2) &= \alpha \ln(c_1 - \tilde{c}_1) + (1 - \alpha) \ln(c_2 - \tilde{c}_2), \quad \tilde{c}_1, \tilde{c}_2 \geq 0 \\ c_1 &= A_1 L_1^{\theta_1}, \quad 0 < \theta_1 \leq 1 \\ c_2 &= A_2 L_2^{\theta_2}, \quad 0 < \theta_2 \leq 1 \\ L_1 + L_2 &= L \end{aligned}$$

These preferences reduce to Cobb-Douglas when $\tilde{c}_1 = \tilde{c}_2 = 0$. Assuming interior solution, we have

$$\begin{aligned} MRS &= MRT \\ \left(\frac{\alpha}{1-\alpha} \right) \frac{c_2 - \tilde{c}_2}{c_1 - \tilde{c}_1} &= \frac{\theta_2 A_2 L_2^{\theta_2-1}}{\theta_1 A_1 L_1^{\theta_1-1}} \\ \left(\frac{\alpha}{1-\alpha} \right) \frac{A_2 (L - L_1)^{\theta_2} - \tilde{c}_2}{A_1 L_1^{\theta_1} - \tilde{c}_1} &= \left(\frac{\theta_2}{\theta_1} \right) \frac{A_2 (L - L_1)^{\theta_2-1}}{A_1 L_1^{\theta_1-1}} \\ \left(\frac{\alpha}{1-\alpha} \right) \frac{A_2 (L - L_1)^{\theta_2} - \tilde{c}_2}{A_2 (L - L_1)^{\theta_2-1}} &= \left(\frac{\theta_2}{\theta_1} \right) \frac{A_1 L_1^{\theta_1} - \tilde{c}_1}{A_1 L_1^{\theta_1-1}} \\ \left(\frac{\alpha}{1-\alpha} \right) \left[L - L_1 - \frac{\tilde{c}_2 (L - L_1)^{1-\theta_2}}{A_2} \right] &= \left(\frac{\theta_2}{\theta_1} \right) \left[L_1 - \frac{\tilde{c}_1 L_1^{1-\theta_1}}{A_1} \right] \\ \left(\frac{\alpha}{1-\alpha} \right) L - \left(\frac{\alpha}{1-\alpha} \right) L_1 - \left(\frac{\alpha}{1-\alpha} \right) \frac{\tilde{c}_2 (L - L_1)^{1-\theta_2}}{A_2} &= \left(\frac{\theta_2}{\theta_1} \right) L_1 - \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1 L_1^{1-\theta_1}}{A_1} \\ \left(\frac{\alpha}{1-\alpha} \right) L + \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} L_1^{1-\theta_1} - \left(\frac{\alpha}{1-\alpha} \right) \frac{\tilde{c}_2}{A_2} (L - L_1)^{1-\theta_2} - L_1 \left[\frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} \right] &= 0 \end{aligned}$$

Let

$$\begin{aligned} \Psi(L_1, L) &= \left(\frac{\alpha}{1-\alpha} \right) L + \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} L_1^{1-\theta_1} - \left(\frac{\alpha}{1-\alpha} \right) \frac{\tilde{c}_2}{A_2} (L - L_1)^{1-\theta_2} - \phi L_1 \\ \text{where } \phi &= \frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} > 0 \end{aligned}$$

We showed in the previous section that $\Psi_1 < 0$ always. Next, we show that the sign of Ψ_2 is always positive.

$$\Psi_2 = \frac{\alpha}{1-\alpha} - (1-\theta_2) \left(\frac{\alpha}{1-\alpha} \right) \frac{\tilde{c}_2}{c_2}$$

The lower bound of Ψ_2 is attained when $c_2 \searrow \tilde{c}_2$, hence

$$\Psi_2 < \frac{\alpha}{1-\alpha} - (1-\theta_2) \left(\frac{\alpha}{1-\alpha} \right) = \frac{\theta_2 \alpha}{1-\alpha}$$

which is always positive. To summarize,

$$\frac{\partial L_1}{\partial L} = - \frac{\overset{(+)}{\Psi_2}}{\underset{(-)}{\Psi_1}} > 0$$

This means that an increase in labor endowment will increase the labor allocation to sector 1, and by symmetry to sector 2 as well.

More important is to find out what happens to the fraction of labor employed in each sector, when total labor endowment goes up.

$$\begin{aligned} \left(\frac{\alpha}{1-\alpha} \right) L + \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} L_1^{1-\theta_1} - \left(\frac{\alpha}{1-\alpha} \right) \frac{\tilde{c}_2}{A_2} (L-L_1)^{1-\theta_2} - L_1 \left[\frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} \right] &= 0 \\ \frac{\alpha}{1-\alpha} + \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} \frac{L_1^{1-\theta_1}}{L} - \left(\frac{\alpha}{1-\alpha} \right) \frac{\tilde{c}_2}{A_2} \frac{(L-L_1)^{1-\theta_2}}{L} - \frac{L_1}{L} \left[\frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} \right] &= 0 \\ \frac{\alpha}{1-\alpha} + \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} \frac{L^{1-\theta_1} \lambda^{1-\theta_1}}{L} - \left(\frac{\alpha}{1-\alpha} \right) \frac{\tilde{c}_2}{A_2} \frac{L^{1-\theta_2} (1-\lambda)^{1-\theta_2}}{L} - \lambda \left[\frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} \right] &= 0 \\ \frac{\alpha}{1-\alpha} + \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} L^{-\theta_1} \lambda^{1-\theta_1} - \left(\frac{\alpha}{1-\alpha} \right) \frac{\tilde{c}_2}{A_2} L^{-\theta_2} (1-\lambda)^{1-\theta_2} - \lambda \left[\frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} \right] &= 0 \end{aligned}$$

Suppose that $\tilde{c}_2 = 0$. Then

$$\Psi(\lambda, L) = \frac{\alpha}{1-\alpha} + \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} L^{-\theta_1} \lambda^{1-\theta_1} - \lambda \left[\frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} \right] = 0$$

$$\begin{aligned} \Psi_1 &= (1-\theta_1) \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} L^{-\theta_1} \lambda^{1-\theta_1} - \left[\frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} \right] \\ \lambda \Psi_1 &= (1-\theta_1) \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} L^{-\theta_1} \lambda^{1-\theta_1} - \lambda \left[\frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} \right] \\ &= \underbrace{\frac{\alpha}{1-\alpha} + \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} L^{-\theta_1} \lambda^{1-\theta_1} - \lambda \left[\frac{\alpha}{1-\alpha} + \frac{\theta_2}{\theta_1} \right]}_{\Psi(\lambda, L)=0} - \theta_1 \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} L^{-\theta_1} \lambda^{1-\theta_1} - \frac{\alpha}{1-\alpha} < 0 \end{aligned}$$

$$\Psi_2 = -\theta_1 \left(\frac{\theta_2}{\theta_1} \right) \frac{\tilde{c}_1}{A_1} L^{-\theta_1-1} \lambda^{1-\theta_1} < 0$$

If at least one sector does not have minimum consumption requirement, then we have the following results

$$\text{sign}(\partial \lambda / \partial L) \begin{cases} > 0 & \text{if } \tilde{c}_2 > 0 \text{ and } \tilde{c}_1 = 0 \\ = 0 & \text{if } \tilde{c}_2 = 0 \text{ and } \tilde{c}_1 = 0 \\ < 0 & \text{if } \tilde{c}_2 = 0 \text{ and } \tilde{c}_1 > 0 \end{cases}$$

It is more difficult to find out what happens to λ when both $\tilde{c}_2 > 0$ and $\tilde{c}_1 > 0$.

8.2.3 Quasi-linear utility, Cobb-Douglas production

Suppose that

$$\begin{aligned} u(c_1, c_2) &= \gamma \ln(c_1) + c_2 \\ c_1 &= A_1 L_1^{\theta_1}, \quad 0 < \theta_1 \leq 1 \\ c_2 &= A_2 L_2^{\theta_2}, \quad 0 < \theta_2 \leq 1 \\ L_1 + L_2 &= L \end{aligned}$$

Assuming interior solution, we have

$$\begin{aligned} MRS &= MRT \\ \frac{\gamma}{c_1} &= \frac{\theta_2 A_2 L_2^{\theta_2-1}}{\theta_1 A_1 L_1^{\theta_1-1}} \\ \frac{\gamma}{A_1 L_1^{\theta_1}} &= \frac{\theta_2 A_2 (L - L_1)^{\theta_2-1}}{\theta_1 A_1 L_1^{\theta_1-1}} \\ \frac{(L - L_1)^{1-\theta_2}}{L_1} &= \frac{\theta_2 A_2}{\theta_1 \gamma} \\ \Psi(L_1, L) &= \frac{(L - L_1)^{1-\theta_2}}{L_1} - \frac{\theta_2 A_2}{\theta_1 \gamma} \end{aligned}$$

It is clear without taking derivatives that

$$\frac{\partial L_1}{\partial L} = -\frac{\Psi_2}{\Psi_1} > 0$$

This means that an increase in the labor endowment will increase the labor allocated to sector 1. Similarly, it is easy to show that $\partial L_2 / \partial L > 0$.

$$\begin{aligned} \Psi(L_2, L) &= \frac{L_2^{1-\theta_2}}{L - L_2} \\ \frac{\partial L_2}{\partial L} &= -\frac{\Psi_2}{\Psi_1} > 0 \end{aligned}$$

Thus, an increase in labor endowment increases the labor employed in each sector. More interestingly, we would like to know what happens to the *fraction* of labor employed in each sector as a result of an increase in total labor endowment.

$$\begin{aligned} \Psi(\lambda, L) &= \frac{(L - \lambda L)^{1-\theta_2}}{\lambda L} - \frac{\theta_2 A_2}{\theta_1 \gamma} \\ &= \frac{(1 - \lambda)^{1-\theta_2}}{\lambda L^{\theta_2}} - \frac{\theta_2 A_2}{\theta_1 \gamma} \\ \frac{\partial \lambda}{\partial L} &= -\frac{\Psi_2}{\Psi_1} \end{aligned}$$

Clearly $\Psi_1 < 0$ and $\Psi_2 < 0$, thus

$$\frac{\partial \lambda}{\partial L} < 0$$

This means that higher labor endowment will push the labor towards the sector that produces the good with constant marginal utility.

References

- [1] Stokey, Nancy L, Robert E. Lucas and Edward C Prescott (1989), *Recursive Methods in Economic Dynamics*, Harvard University Press
- [2] **Hansen, Gary D. and Prescott, Edward C (2003)**, “Malthus to Solow.” *American Economic Review*, 92(4), pp. 1205-1217.
- [3] **Golosov, Mikhail; Jones, Larry E. and Tertilt, Michele (2006)**, “Efficiency with Endogenous Population Growth.” Forthcoming in *Econometrica*.